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Almost Graded Prime Ideals

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Abstract: Problem Statement: Graded commutative ring with unity over an abelian group were introduced by many authors such as T. Y. Lam and C. T. C. Wall, and almost prime ideals over commutative rings with unity were introduced by S.M. Batwadeker and P.K. Sharma, and this forced us to try to extend the theory of almost and n-almost prime ideals to the graded case. **Approach:** We develop the theory of almost and n-almost prime ideals to the graded case. **Results:** We extended some basic results about almost and n-almost prime ideals to the graded case, and then we gave a relationship between n-almost graded prime ideals and weakly graded prime ideals. **Conclusion:** The extended results about almost and n-almost graded prime ideals allow us to classify further properties about almost graded prime ideals. 2000 AMS Mathematics Subject Classification: 13 A 02.

Key words: Graded rings, graded prime ideals, almost graded prime ideals, n-almost graded prime ideals.

INTRODUCTION

Almost prime ideals in commutative rings with $1 \neq 0$ have been introduced by[4].

In this study we classify almost graded prime ideals in G-graded Commutative ring R with $1 \neq 0 \in$ R, where G is an abelian group, also we do consider some various properties of almost graded prime ideals.

Let G be an abelian group with identity $e \in G$. By G-graded Commutative ring we mean a commutative ring R with $1 \neq 0 \in$ R together with a direct sum (as an additive group) $R = \oplus_{g\in G} R_g$ with the property $R_g R_h \subseteq R_{gh} \forall g, h \in G$. The Summands $R_g, g \in G$, are homogeneous Components and elements of R_g are called homogeneous elements, we also write $h(R) = \bigcup_{g\in G} R_g$. Moreover, if $R = \oplus_{g\in G} R_g$ is a G-graded ring with identity $0 \neq 1 \in$ R, then R_e is a subring of R with $0 \neq 1 \in R_e$ and R_g is an R_e-module for all $g \in G$.

Let $R = \oplus_{g\in G} R_g$ be a G-graded ring with $0 \neq 1 \in R_e$, a (right) R-graded module M is an R-module with grading $M = \oplus_{g\in G} M_g$ as (right) R_e-module such that $m_g, r_h \in M_{gh}$ for any $m_g \in M_g, r_h \in R_h, g, h \in G$. A (right) G-graded ideal $I = \oplus_{g\in G} I_g$ is a (right) R-graded module of the G-graded ring R.

Clearly {0} is a graded ideal, also if I, J are G-graded ideals of R then:

$$(I : J) = \{a \in R : aJ \subseteq I\}$$

is a G-graded ideal[5].

A G-graded ideal I of R is said to be graded prime ideal if $I \neq R$ and whenever $a_g b_h \in I_{gh}$ we have $a_g \in I_g$ or $b_h \in I_h$ and I is called weakly graded prime ideal if $I \neq R$ and whenever $0 \neq a_g b_h \in I_{gh}$ then either $a_g \in I_g$ or $b_h \in I_h$[3].

In this study we prove that a graded ideal I is an almost graded prime ideal if and only if $S = \bigcup_{g\in G}(R_g - I_g)$ is almost multiplicatively closed. Moreover, we define an n-almost graded prime ideals for $n \geq 2$ such that 2-almost graded prime ideals are almost graded prime ideals and we prove that I is an n-almost graded prime ideal if and only if $(I_h : x_g) = I_{hg^{-1}}$ or $(I_h : x_g) = (I^n \bigcap R_h : x_g)$.

Finally, we prove that I is an n-almost graded prime ideal in R if and only if I/I^n is weakly graded prime ideal in R/I^n.

MATERIALS AND METHODS

We generalize some basic definitions and lemmas about almost and n-almost prime ideals to the graded case.

Definition 1: Let R be commutative G-graded ring, where G is an abelian group.

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A proper graded ideal P of R is said to be almost graded prime ideal if:

$$a_g b_h \in P_{gh} - [P^2 \cap R_{gh}]$$

implies $a_g \in P_g$ or $b_h \in P_h$.

Example: Let $R = [i]$ be a Z_2-graded ring and let $R_{\bar{0}} = Z, R_{\bar{1}} = iZ$. Then R is commutative Z_2-graded ring with $1 \in R_{\bar{0}}$. One can easily check that the graded ideal:

$$P = 2R = \{2a + 2bi : a, b \in Z\}$$

With $P_{\bar{0}} = 2Z$ and $P_{\bar{1}} = 2Zi$ is an almost graded prime ideal in R, but P as an ideal in R is not almost prime ideal, since $(1-i)(1+i) = 2 \in P - P^2$ and $1-i, 1+i \notin P$.

Lemma 1: Le P, A, B be arbitrary graded ideals of a commutative graded ring R. If $P_g \subseteq A_g \cup B_g$, then $P_g \subseteq A_g$ or $P_g \subseteq B_g$. In particular, if $P_g = A_g \cup B_g$, then $P_g = A_g$ or $P_g = B_g$.

Proof: Assume that $P_g \subseteq A_g \cup B_g$, but $P_g \not\subset A_g$ and $P_g \not\subset B_g$. So we have $a_g, b_g \in P_g$ where $a_g \notin B_g$ and $b_g \notin A_g$, therefore $a_g \in A_g$ and $b_g \in B_g$. But:

$$a_g + b_g \in P_g \subseteq A_g \cup B_g$$

so $a_g + b_g \in A_g$ or $a_g + b_g \in B_g$. In the first case $b_g \in A_g$ while the second case $a_g \in B_g$, both contradictions. The second statement follows because $P_g = A_g \cup B_g$ implies $A_g \subseteq P_g$ or $B_g \subseteq P_g$.

Lemma 2: Let R be commutative G-graded ring, let I be a graded ideal in R and let $x_g \notin I_g$ for some g ∈ G. Then (I : x_g) is a G-graded ideal in R, Where:

$$(I : x_g) = \{r \in R : rx_g \in I\}$$

Proof: First of all we show that $(I : x_g) = \oplus_{h\in G}(I_h : x_g)$. One can easily see that $\oplus_{h\in G}(I_h : x_g) \subseteq (I : x_g)$. Conversely, let $r \in (I : x_g)$, then $rx_g \in I$ which implies that $r_h x_g \in I_{hg}$ for all h ∈ G, thus $r_h \in (I_{hg} : x_g)$ for all h∈G. So:

$$r = \sum r_{h'} \in (I_{h'g} : x_g) = \oplus_{h\in G}(I_h : x_g)$$

Therefore:

$$(I : x_g) = \oplus_{h\in G}(I_h : x_g)$$

Next we show that (I: x_g) is a G-graded ideal. It is easy to see that (I_h: x_g) is closed under addition. Now let $a_h \in (I_{hg} : x_g)$ and let $r_{h'} \in R_{h'}$, then $r_{h'} a_h x_g \in r_{h'} I_{hg} \subseteq I_{h'hg}$, therefore $r_{h'} a_h \in (I_{h'hg} : x_g) \subseteq (I : x_g)$. Thus (I: x_g) is a G-graded ideal.

We can also prove the lemma above in another way, by showing that (I: x_g) = (I: Rx_g), where $Rx_g = \{rx_g : r \in R\}$ is a G-graded ideal in R, so by[4] (I: x_g) is a G-graded ideal.

Now by the lemma above if I is a graded ideal in a is a G-graded ring R and $x_g \notin I_g$ for some g ∈ G then (I: x_g) and (I^2: x_g) are graded ideals in R. Thus we have the following result about almost graded prime ideals.

Theorem 1: For a proper graded ideal I of a G-graded ring R the following assertions are equivalent.

a) I is almost graded prime ideal.
b) For $x_g \in R_g - I_g, (I_h : x_g) = I_{hg^{-1}} \cup (I^2 \cap R_h : x_g)$
c) For $x_g \in R_g - I_g$, $(I_h : x_g) = I_{hg^{-1}}$ or $(I_h : x_g) = (I^2 \cap R_h : x_g)$
d) For R_e-modules $A_g \subseteq R_g$ and $B_h \subseteq R_h$ with $A_g B_h \subseteq I_{gh}$, but $A_g B_h \not\subset I^2 \cap R_{gh}$, then $A_g \subseteq I_g$ or $B_h \subseteq I_h$

Proof:

- (a) ⇒ (b) Let $y_{hg^{-1}} \in (I_h : x_g)$, then $y_{hg^{-1}} x_g \in I_h$, if $y_{hg^{-1}} x_g \notin I^2 \cap R_h$, then $y_{hg^{-1}} x_g \in I_h - (I^2 \cap R_h)$, but I is almost graded prime ideal, so $y_{hg^{-1}} x_g \in I_{hg^{-1}}$ since $x_g \notin I_g$, but if $y_{hg^{-1}} x_g \in I^2 \cap R_h$, then $y_{hg^{-1}} \in (I^2 \cap R_h : x_g)$, so $(I_h : x_g) \subseteq I_{hg^{-1}} \subset (I^2 \cap R_h : x_g)$, but $I^2 \cap R_h \subseteq I_h$ and $I_{hg^{-1}} \subseteq (I_h : x_g)$ thus $I_{hg^{-1}} \cup (I^2 \cap R_h : x_g) \subseteq (I_h : x_g)$ and hence

$$(I_h : x_g) = I_{hg^{-1}} \cup (I^2 \cap R_h : x_g)$$

- (b) $\Rightarrow$ (c) Follows from Lemma 1 and 2
- (c) $\Rightarrow$ (d) Let $A_g \subseteq R_g$, $B_h \subseteq R_h$ with $A_gB_h \subseteq I_{gh}$, but $A_gB_h \not\subset I^2 \bigcap R_{gh}$. We show that $A_g \subseteq I_g$ or $B_h \subseteq I_h$. Suppose on contrary that $A_g \not\subset I_g$ and $B_h \not\subset I_h$, then for $a_g \in A_g - I_g, a_gB_h \subseteq I_{gh}$ which implies that $B_h \subseteq (I_{gh} : a_g)$, but since $B_h \not\subset I_h$ we have $(I_{gh} : a_g) \neq I_h$. Thus by part (c) of the theorem $(I_{gh} : a_g) = (I^2 \bigcap R_h : a_g)$ and so $B_h \subseteq (I^2 \bigcap R_{gh} : a_g)$. Hence $a_gB_h \subset I^2 \bigcap R_{gh}$. Similarly, if $b_h \in B_h - I_h$, then $A_gb_h \subseteq I^2 \bigcap R_{gh}$. Finally, for any $a_g \in A_g \bigcap I_g$ and $b_h \in B_h \bigcap I_h$ then $a_gb_h \in I^2 \bigcap R_{gh}$. Therefore $A_gb_h \subseteq I^2 \bigcap R_{gh}$, a contradiction. Thus $A_g \subseteq R_g$ or $B_h \subseteq I_h$
- (d) $\Rightarrow$ (a) Let $a_gb_h \in I_{gh} - I^2 \bigcap R_{gh}$ and let $A_g = a_gR_e, B_h = b_hR_e$, where e is the identity in G. Then $A_gB_h \subseteq I_{gh}$, but $A_gB_h \not\subset I^2 \bigcap R_{gh}$ so by part (d) of the theorem $A_g \subseteq I_g$ or $B_h \subseteq I_h$ and this implies that $a_g \in I_g$ or $b_h \in I_h$ which means that I is almost graded prime ideal

Definition 2: Let R be a G-graded ring. Then a nonempty subset S of h(R) is called graded–multiplicatively closed whenever a, b$\in$S then ab$\in$S.

Lemma 3: A grade ideal I of a G-graded ring R is a graded prime ideal if and only if $S = \bigcup_{g\in G}(R_g - I_g)$ is graded-multiplicatively closed.

Proof: ($\Rightarrow$) Suppose I is a graded prime ideal and let x, y $\in$ S, then $x = a_g \in R_g - I_g, y = b_h \in R_h - I_h$ for some g, h $\in$ G. Now $xy = a_gb_h \in S$, if not then $xy = a_gb_h \notin R_{gh} - I_{gh}$, which means that $xy = a_gb_h \in I_{gh}$, since I is graded prime ideal $a_g \in I_g$ or $b_h \in I_h$ a contradiction. Hence S is graded–multiplicatively closed.

($\Leftarrow$) Suppose S is graded–multiplicatively closed and let $a_gb_h \in I_{gh}$, then $a_gb_h \notin R_{gh} - I_{gh}$ which means that $a_gb_h \notin S$ and so $a_g \notin R_g - I_g$ or $b_h \notin R_h - I_h$ and this is equivalent to say that $a_g \in I_g$ or $b_h \in I_h$.

Definition 3: Let R be a G-graded ring and let I be a graded ideal of R. Then I is called weakly graded prime ideal of R if $0 \neq a_gb_h \in I_{gh}$, then $a_g \in I_g$ or $b_h \in I_h$.

Definition 4: Let R be a G-graded ring. Then a non empty subset S of h(R) is called weakly graded-multiplicatively closed if a, b$\in$S then ab $\in$ S or ab = 0.

Lemma 4: A graded ideal I of a G-graded ring R is weakly graded prime if and only if $S = \bigcup_{g\in G}(R_g - I_g)$ is weakly graded-multiplicatively closed.

Proof: ($\Rightarrow$) Suppose I is weakly graded prime ideal and let x, y $\in$ S, then $x = a_g \in R_g - I_g, y = b_h \in R_h - I_h$ for some g, h $\in$ G, thus $a_g \notin I_g$ and $b_h \notin I_h$. If $xy = a_gb_h \neq 0$, then $a_gb_h \notin I_{gh}$, since I is weakly graded prime ideal, which implies that $xy = a_gb_h \in R_{gh} - I_{gh} \subseteq S$. Thus xy = 0 or xy $\in$ S. Hence S is weakly graded-multiplicatively closed.

($\Leftarrow$) Suppose S is weakly graded–multiplicatively closed and let $0 \neq a_gb_h \in I_{gh}$, then $a_gb_h \notin R_{gh} - I_{gh}$, so $a_gb_h \notin S$ which implies $a_g \notin R_g - I_g$ or $b_h \notin R_h - I_h$. Thus $a_g \in I_g$ or $b_h \in I_h$, hence I is weakly graded prime ideal.

Definition 5: If $A \neq \Phi$ is a subset of a graded ring R, then we define A^2 to be the smallest graded ideal in R Containing $\{ab : a,b \in A\}$.

Lemma 5: Let R be a G-graded ring and let I be a graded ideal in R. If:

$$S = \bigcup_{g\in G}(R_g - I_g)$$

Then:

(a) $(h(R) - S) \bigcap R_h = I_h$ for all $h \in G$

(b) $(h(R) - S)^2 = I^2$

Proof:

(a) Let $x_h \in I_h$ then $x_h \notin R_h - I_h$ which means $x_h \notin S$, thus $x_h \in (h(R) - S) \bigcap R_h$. Conversely, let $x_h \in (h(R) - S) \bigcap R_h$, then $x_h \notin S \bigcap R_h$, hence $x_h \notin R_h - I_h$ which means that $x_h \in I_h$.

(b) $(h(R) - S)^2 = I^2$ follows from (a)

Definition 6: Let R be a G-graded ring. Then a nonempty subset of R is called almost graded-multiplicatively closed if a, b $\in$ S implies:

$$ab \in S \text{ or } ab \in (h(R) - S)^2$$

Lemma 6: A graded ideal I of a G-graded ring R is almost graded prime ideal if and only if

$S = \bigcup_{g\in G}(R_g - I_g)$ is almost graded-multiplicatively closed.

Proof: ($\Rightarrow$) Suppose that I is almost graded prime ideal and let x, y $\in$ S, then $x = a_g \in R_g - I_g, y = b_h \in R_h - I_h$ for some g, h $\in$ G. So $a_g \notin I_g$ and $b_h \notin I_h$, hence $a_gb_h \notin I_{gh} - (I^2 \cap R_{gh})$. Therefore $a_gb_h \notin I_{gh}$ or $a_gb_h \in (I^2 \cap R_{gh})$ which implies that $a_gb_h \in R_{gh} - I_{gh} \subseteq S$ or $a_gb_h \in I^2 \cap R_{gh}$, but by Lemma 5, $(h(R) - S)^2 = I^2$, where $h(R) = \bigcup_{g\in G} R_g$, so $a_gb_h \in S$ or $a_gb_h \in (h(R) - S)^2$, thus S is almost graded-multiplicatively closed.

($\Leftarrow$) Suppose S is an almost graded-multiplicatively closed and let $a_gb_h \in I_{gh} - (I^2 \cap R_{gh})$. Then $a_gb_h \notin I^2 \cap R_{gh}$, we want to show that $a_g \in I_g$ or $b_h \in I_h$. Suppose on contrary that $a_g \in R_g - I_g$ and $b_h \in R_h - I_h$, then a_g and b_h are in S and since S is almost graded-multiplicatively closed, $a_gb_h \in S$ or $a_gb_h \in (h(R) - S)^2$. But $a_gb_h \in I_{gh}$ which means that $a_gb_h \notin S$. So $a_gb_h \in (h(R) - S)^2$, but by Lemma 5, $(h(R) - S)^2 = I^2$ which means that $a_gb_h \in I^2 \cap R_{gh}$ contradiction. Therefore $a_g \in I_g$ or $b_h \in I_h$ and hence I is almost graded prime ideal.

Definition 7: Let R be a commutative G-graded ring, where G is an abelian group. A proper graded ideal P of R is said to be n-almost graded prime ideal if:

$$a_gb_h \in P_{gh} - [P^n \cap R_{gh}]$$

implies $a_g \in P_g$ or $b_h \in P_h$.

RESULTS AND DISCUSSION

By Lemma 2 if I is a graded ideal in a G-graded ring R and $x_g \notin I_g$ for some $g \in G$ then $(I : x_g)$ and $(I^n : x_g)$ are graded ideals in R. So 2-almost graded prime ideals are the same as almost graded prime ideals. The following result will characterize n-almost graded prime ideals.

Theorem 2: For a proper graded ideal I of a graded ring R the following assertions are equivalent.

(a) I is n-almost graded prime ideal
(b) For $x_g \in R_g - I_g, (I_h : x_g) = I_{hg^{-1}} \cup (I^n \cap R_h : x_g)$.
(c) For $x_g \in R_g - I_g, (I_h : x_g) = I_{hg^{-1}}$ or $(I_h : x_g) = (I^n \cap R_h : x_g)$
(d) For R_e-modules $A_g \subseteq I_g$ and $B_h \subseteq I_h$ with $A_gB_h \subseteq I_{gh}$, but $A_gB_h \not\subset I^n \cap R_{gh}$, then $A_g \subseteq I_g$ or $B_h \subseteq I_h$

Proof:

- (a) $\Rightarrow$ (b) Let $y_{hg^{-1}} \in (I_h : x_g)$, then $y_{hg^{-1}}x_g \in I_h$, if $y_{hg^{-1}}x_g \in I^n \cap R_h$, then $y_{hg^{-1}} \in (I^n \cap R_h : x_g)$, while if $y_{hg^{-1}}x_g \notin I^n \cap R_h$, then $y_{hg^{-1}}x_g \in I_h - (I^n \cap R_h)$ and since I is n-almost graded prime ideal and $x_g \notin I_g$, then $y_{hg^{-1}} \in I_{hg^{-1}}$, so $(I_h : x_g) \subseteq I_{hg^{-1}} \cup (I^n \cap R_h : x_g)$, but $I^n \cap R_h \subseteq I_h$ and $I_{hg^{-1}} \subseteq (I_h : x_g)$, so $I_{hg^{-1}} \cup (I^n \cap R_h : x_g) \subseteq (I_h : x_g)$ and hence $(I_h : x_g) = I_{hg^{-1}} \cup (I^n \cap R_h : x_g)$.
- (b) $\Rightarrow$ (c) Since $(I^n \cap x_g)$ and I are G-graded ideals and:

$$(I_h : x_g) = I_{hg^{-1}} \cup (I^n \cap R_h : x_g)$$

then by Lemma 1:

$$(I_h : x_g) = I_{hg^{-1}} \text{ or } (I_h : x_g) = (I^n \cap R_h : x_g)$$

- (c) $\Rightarrow$ (d) Let $A_g \subseteq I_g$, $B_h \subseteq I_h$ with $A_gB_h \subseteq I_{gh}$, but $A_gB_h \not\subset I^n \cap R_{gh}$. We show that $A_g \subseteq I_g$ or $B_h \subseteq I_h$. Suppose on contrary that $A_g \not\subset I_g$ and $B_h \not\subset I_h$, then let $a_g \in A_g - I_g, a_gB_h \subseteq I_{gh}$ and hence $B_h \subseteq (I_{gh} : a_g)$, but $B_h \not\subset I_h$, so by part (c) of the theorem $(I_{gh} : a_g) = (I^n \cap R_{gh} : a_g)$ and therefore $B_h \subseteq (I^n \cap R_{gh} : a_g)$ which implies that $a_gB_h \subset I^n \cap R_{gh}$. Next let $a_g \in A_g \cap I_g$, choose $a'_g \in A_g \cap I_g$, then $a_g + a'_g \in A_g - I_g$ so by the first case a'_gB_h and $(a_g + a'_g)B_h$ are subsets of $I^n \cap R_{gh}$. Thus for any $b_h \in B_h, a'_gb_h$ and $(a_g + a'_g)b_h$ are in $I^n \cap R_{gh}$, so $a_gb_h \in I^n \cap R_{gh}$ and hence $A_gB_h \subseteq I^n \cap R_{gh}$ a contradiction. Therefore $A_g \subseteq I_g$ or $B_h \subseteq I_h$.
- (d) $\Rightarrow$ (a) Let $a_gb_h \in I_{gh} - I^n \cap R_{gh}$ and let $A_g = a_gR_e, B_h = b_hR_e$, where e is the identity in G.

Then $A_gB_h \subseteq I_{gh}$, but $A_gB_h \not\subset I^n \cap R_{gh}$ so by part (d) of the theorem $A_g \subseteq I_g$ or $B_h \subseteq I_h$ and this implies that $a_g \in I_g$ or $b_h \in P_h$ which means that I is n-almost graded prime ideal

The following result shows the relationship between n-almost graded prime ideals and weakly prime ideals.

Theorem 3: Let R be a commutative G-graded ring, then a graded ideal Iof R is an n-almost graded prime ideal if and only if I/I^n is weakly graded prime ideal in the G-graded ring R/I^n.

Proof: Suppose I is an n-almost graded prime ideal in R and let:

$$0 \neq (a_g + I^n)(b_h + I^n) \in (I/I^n)_{gh}$$

then $0 \neq a_g b_h + I^n \in (I/I^n)_{gh}$ which implies that $a_g b_h \in I_{gh}$ and $a_g b_h \notin I^n \cap R_{gh}$ and hence $a_g \in I_g$ or $b_h \in I_h$, since I is an n-almost graded prime ideal in R. Thus $a_g + I^n \in (I/I^n)_g$ or $b_h + I^n \in (I/I^n)_h$ which implies that I/I^n is weakly graded prime ideal in R/I^n.
Conversely, suppose that I/I^n is weakly graded prime ideal in R/I^n and let:

$$a_g b_h \in I_{gh} - [I^n \cap R_{gh}]$$

Then:

$$(a_g + I^n)(b_h + I^n) = a_g b_h + I^n \neq 0 \in (I/I^n)_{gh}$$

Since I/I^n is weakly graded prime ideal in R/I^n then $a_g + I^n \in (I/I^n)_g$ or $b_h + I^n \in (I/I^n)_h$ which implies that $a_g \in I_g$ or $b_h \in I_h$, thus I is an n-almost graded prime ideal in R.

CONCLUSION

The developing theory of almost and n-almost prime ideals were extended smoothly to the graded case, and the extension of some basic results (Theorem 2 and Theorem 3) would help us to classify further properties about almost and n-almost graded prime ideals.

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