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Stability Analysis of a Class of Three-Neuron Delayed Cellular Neural Network

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Abstract: Problem statement: In this study linear stability of a class of three neuron cellular network with transmission delay had been studied. **Approach:** The model for the problem was first presented. The problem is then formulated analytically and numerical simulations pertaining to the model are carried out. **Results:** A necessary and sufficient condition for asymptotic stability of trivial steady state in the absence of delay is derived. Then a delay dependent sufficient condition for local asymptotic stability of trivial, steady state and sufficient condition for no stability switching of trivial steady for such a network are derived. Numerical simulation results of the model were presented. **Conclusion/Recommendations:** From numerical simulation, it appears that there may be a possibility of multiple steady states of the model. It may be possible to investigate the condition for the existence of periodic solutions of the non-linear model analytically.

Key words: Cellular neural network, asymptotic stability, delay dependent stability, delay independent stability switching

INDRODUCTION

The notion of Cellular Neural Network (CNNs) was introduced by Chua and Yang (1998) and since then, CNN models have been used in many engineering applications, e.g., in signal processing and especially in static image treatment (Chua, 1988). CNN is used in various type of motion related applications such as processing of moving images, speed detection of moving objects, pattern classification. In order to achieve these tasks a delay parameter was introduced into the CNN system equations. Arik and Tavsanoglu (1998) studied the global asymptotic stability and exponential stability of Delayed Neural Network (DCNN). Periodic solutions and exponential stability in delayed cellular network and sufficient conditions for global asymptotic stability of cellular neural network with delay are discussed by Cao (2000); Zhang *et al*. (2007) and others (Gyori and Hartung, 2004) respectively.

In this study we have considered a class of three- neuron delayed cellular neural network and have studied the local stability phenomenon of its trivial equilibrium. First the model is represented and linear analysis is done. The necessary and sufficient condition of local asymptotic stability of the trivial steady state (0,0,0) in absence of time delay has been derived. Then length of time delay has been estimated, below which trivial steady state remains asymptotically stable. That is delay dependent condition of local asymptotic stability of (0.0,0) is derived. Next sufficient condition for no stability switching of (0,0,0) has been derived considering that there is no self - connection. Numerical simulation of the model, that confirms the results obtained analytically, is also presented. Finally a conclusion has been drawn.

MATERIALS AND METHODS

Model description: In this study we have considered here a three neuron cellular network with discrete delay described by following system of delay differential equations:

$$\begin{aligned}
\frac{dx_1}{dt} &= -kx_1(t) + a_{11}\tanh[x_1(t)] + a_{12}\tanh[x_2(t)] \\
&\quad + a_{13}\tanh[x_3(t)] + b_{11}\tanh[x_1(t-\tau)] \\
&\quad + b_{12}\tanh[x_2(t-\tau)] + b_{13}\tanh[x_3(t-\tau)] \\
\frac{dx_2}{dt} &= -kx_2(t) + a_{21}\tanh[x_1(t)] + a_{22}\tanh[x_2(t)] \\
&\quad + a_{23}\tanh[x_3(t)] + b_{21}\tanh[x_1(t-\tau)] \\
&\quad + b_{22}\tanh[x_2(t-\tau)] + b_{23}\tanh[x_3(t-\tau)] \\
\frac{dx_3}{dt} &= -kx_3(t) + a_{31}\tanh[x_1(t)] + a_{32}\tanh[x_2(t)] \\
&\quad + a_{33}\tanh[x_3(t)] + b_{31}\tanh[x_1(t-\tau)] \\
&\quad + b_{32}\tanh[x_2(t-\tau)] + b_{33}\tanh[x_3(t-\tau)]
\end{aligned} \tag{1}$$

Here x_i (t) (i = 1, 2, 3) corresponds to the activation state of the neuron 'i' at time 't'. k>0 is the decay rate of neurons, that is, it represents the rate with which a neuron will reset to its potential to the resisting state in isolation when disconnected from network.a_{ij} and b_{ij} are the weights of synaptic connections from neuron 'j' to neuron 'i' at time 't' and (t-τ) respectively. τ≥0 corresponds to the

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transmission delay along the axon. It is assumed that system (1) is provided with initial conditions:

$$x_i(s) = \phi_i(s)(i = 1,2,3)\ s\varepsilon[-\tau_i,0]$$

where, ϕ_i: $[-\tau_i,0]\rightarrow R$ is assumed to be continued.

Due to the complexity of system (1), a special case of general model has been considered to make the calculation tractable. For time 't' and (t-τ) the synaptic weights due to self-connections are taken as γ and β respectively. Other synaptic weights areas follows:

$$a_{ii} = \gamma;\ b_{ii} = \beta$$
$$a_{12} = a_{23} = a_{31} = a;\ b_{12} = b_{23} = b_{31} = b;$$
$$a_{21} = a_{32} = a_{13} = -a;\ b_{21} = b_{32} = b_{13} = -b;$$

Now (1) takes the form:

$$\begin{aligned}
\frac{dx_1}{dt} &= -kx_1(t) + \gamma\tanh[x_1(t)] + a\tanh[x_2(t)] - a\tanh[x_3(t)] \\
&\quad + \beta\tanh[x_1(t-\tau)] + b\tanh[x_2(t-\tau)] - b\tanh[x_3(t-\tau)] \\
\frac{dx_2}{dt} &= -kx_2(t) - a\tanh[x_1(t)] + \gamma\tanh[x_2(t)] + a\tanh[x_3(t)] \\
&\quad - b\tanh[x_1(t-\tau)] + \beta\tanh[x_2(t-\tau)] + b\tanh[x_3(t-\tau)] \\
\frac{dx_3}{dt} &= -kx_3(t) + a\tanh[x_1(t)] - a\tanh[x_2(t)] + \gamma\tanh[x_3(t)] \\
&\quad + b\tanh[x_1(t-\tau)] - b\tanh[x_2(t-\tau)] + \beta\tanh[x_3(t-\tau)]
\end{aligned} \tag{2}$$

Linear analysis: Evidently (0,0,0) is the trivial steady state of (2). To investigate the local stability of steady state (0,0,0), the system (2) has been linearized equations are obtained:

$$\begin{aligned}
\frac{du_1}{dt} &= -ku_1(t) + \gamma u_1(t) + au_2(t) + au_3(t) + \beta u_1 \\
&\quad (t-\tau) + bu_2(t-\tau) + bu_3(t-\tau) \\
\frac{du_2}{dt} &= -ku_2(t) + au_1(t) + \gamma u_2(t) + au_3(t) + bu_1 \\
&\quad (t-\tau) + \beta u_2(t-\tau) + bu_3(t-\tau) \\
\frac{du_3}{dt} &= -ku_3(t) + au_1(t) + au_2(t) + \gamma u_3(t) + bu_1 \\
&\quad (t-\tau) + bu_2(t-\tau) + \beta u_3(t-\tau)
\end{aligned} \tag{3}$$

Substituting $u_i(t) = c_i e^{\lambda t}$ (i = 1,2,3), where c_i is a constant, the characteristic equation corresponding to system (3) is obtained as follows:

$$\begin{vmatrix}
(-k-\lambda+\gamma+\beta e^{-\lambda t}) & a+be^{-\lambda t} & a+be^{-\lambda t} \\
a+be^{-\lambda t} & \begin{matrix}(-k-\lambda \\ +\gamma+\beta e^{-\lambda t})\end{matrix} & a+be^{-\lambda t} \\
a+be^{-\lambda t} & a+be^{-\lambda t} & \begin{matrix}(-k-\lambda+ \\ \gamma+\beta e^{-\lambda t})\end{matrix}
\end{vmatrix} = 0 \tag{4}$$

$$\Rightarrow \lambda^3 + 3\lambda^2(k_1 - \beta e^{-\lambda t}) + 3\lambda[P - Ae^{-\lambda t} + Ce^{-2\lambda t}]$$
$$+[Q - Re^{-2\lambda t} + Ee^{-2\lambda t} + De^{-3\lambda t}] = 0$$

Where:

$$\begin{aligned}
k_1 &= k - \gamma \\
Q &= k(k_1^2 + 3a^2) \\
P &= (k_1^2 + a^2) \\
R &= 3(\beta P - 2abk_1) \\
A &= 2(\beta k_1 - ab) \\
E &= 3(k_1 C - 2ab\beta) \\
C &= (\beta^2 + b^2) \\
D &= \beta(\beta^2 + 3b^2)
\end{aligned}$$

Now in the absence of delay, that is when τ = 0, characteristic Eq. 4 becomes:

$$\begin{aligned}
&\lambda^3 + 3\lambda^2(k_1-\beta) + 3\lambda[P - A + C] + [Q - R + E + D] = 0 \\
&\Rightarrow \lambda^3 + 3\lambda^2(k_1-\beta) + 3\lambda[(k_1-\beta)^2 + (a+b)^2] \\
&+[(k_1-\beta)^3 + 3a^2(k_1-\beta)] = 0 \\
&\Rightarrow \lambda^3 + 3\lambda^2(k-\gamma-\beta) + 3\lambda[(k-\gamma-\beta)^2 + (a+b)^2] \\
&+[(k-\gamma-\beta)^3 + 3a^2(k-\gamma-\beta)] = 0
\end{aligned} \tag{5}$$

By Routh-Hurwitz criteria all the roots of (5) have negative real parts iff the following conditions hold:

$$\begin{aligned}
&\text{(i)}\ \ k>\gamma+\beta \\
&\text{(ii)}\ \ 4(k_1-\beta)^2+3(a+b)^2>0 \\
&\text{(iii)}\ \ 4(k_1-\beta)^2+3(a+b)^2>0
\end{aligned} \tag{6}$$

Here conditions (ii) and (iii) are obvious. Therefore we can state the following theorem:

Theorem 1: For the system (2) if the decay rate of neurons is greater than the sum of synaptic weights corresponding to self connections for a neuron (that is if $k>\gamma+\beta$), then trivial steady state (0, 0, 0) of system (2) is locally asymptotically stable in absence of delay.

Delay dependent local stability criteria: Now we try to find out the delay dependent criteria of local stability of trivial steady state (0, 0, 0) of system (2).

We are to estimate the range of delay using Nyquist criteria for which the trivial steady state (0, 0, 0) of system (2) remains locally asymptotically stable.

To do this first it has been assumed that conditions in (6) holds throughout the following discussion. By continuity for sufficiently small τ>0 all eigen values of (3) have negative real parts provided that no eigen value with positive real part bifurcates from ∞ as τ increases from zero. It is then possible to use a criterion of Nyquist described below, to estimate the range of τ for which the trivial steady state (0, 0, 0) remains asymptotically stable. To do this the system (3) and a space of real valued continuous function defined on $[-\tau, \infty]$ satisfying initial conditions $u_1(t) = u_2(t) = u_3(t) = 0$ for $t<0$ have been derived.

Let $\bar{u}_1(s), \bar{u}_2(s), \bar{u}_3(s)$ denote the Laplace transform of $u_1(t)$, $u_2(t)$, $u_3(t)$ respectively. Taking Laplace transform of (3) we get:

$$\begin{aligned}
&s\overline{u_1}(s) - \overline{u_1}(0) = -k_1\overline{u_1}(s) + a\overline{u_2}(s) - a\overline{u_3}(s) \\
&+\beta\left\{\overline{u_1}(s) + \delta_1(s)\right\}e^{-s\tau} + b\left\{\overline{u_2}(s) + \delta_2(s)\right\}e^{-s\tau} \\
&-b\left\{\overline{u_3}(s) + \delta_3(s)\right\}e^{-s\tau} \\
&s\overline{u_2}(s) - \overline{u_2}(0) = -k_1\overline{u_2}(s) - a\overline{u_1}(s) - a\overline{u_3}(s) \\
&-b\left\{\overline{u_1}(s) + \delta_1(s)\right\}e^{-s\tau} + \beta\left\{\overline{u_2}(s) + \delta_2(s)\right\}e^{-s\tau} \\
&+b\left\{\overline{u_3}(s) + \delta_3(s)\right\}e^{-s\tau} \\
&s\overline{u_3}(s) - \overline{u_3}(0) = -k_1\overline{u_3}(s) - a\overline{u_1}(s) + a\overline{u_2}(s) \\
&+b\left\{\overline{u_1}(s) + \delta_1(s)\right\}e^{-s\tau} - b\left\{\overline{u_2}(s) + \delta_2(s)\right\}e^{-s\tau} \\
&-\beta\left\{\overline{u_3}(s) + \delta_3(s)\right\}e^{-s\tau}
\end{aligned} \tag{7}$$

Where:

$$k_1 = k - \gamma; \qquad \delta_i(s) = \int_{-\tau}^{0} e^{-s\tau}u_i(t)dt \ (i = 1,2,3) \tag{8}$$

Rearranging the terms in (7) and then eliminating $\overline{u_2}(s)$ and $\overline{u_3}(s)$ we get:

$$\begin{aligned}
f(s)\overline{u_1}(s) = {} & A_1\overline{u_1}(0) + A_2\overline{u_2}(0) + A_3\overline{u_3}(0) \\
& + A_4\delta_1(s) + A_5\delta_2(s) + A_6\delta_3(s)
\end{aligned} \tag{9}$$

Where:

$$\begin{aligned}
&F(s) = R^3 + 3RA^2 \\
&A_5 = \left\{A(R+A)(b+\beta) + b(R-A)(A+R)\right\}e^{-s\tau} \\
&A_1 = R^2 + A^2 \\
&A_6 = -(Rb + A\beta)(R-A)e^{-s\tau} \\
&A_2 = A(R+A) \\
&A_3 = -A(R-A) \\
&A_4 = \left\{A(R+A)(\beta-b) + (R-A)(R\beta - Ab)\right\}e^{-s\tau}
\end{aligned}$$

Where R and A are given by:

$$R = (s + k_1 - \beta\, e^{st})$$
$$A = (a + be^{\lambda t})$$

That is, F(s) becomes:

$$F(s) = (s+k_1-\beta e^{\lambda t})^3 + 3(s+k_1-\beta e^{\lambda t})(a+be^{\lambda t})^2 \tag{10}$$

The inverse Laplace transform of $II_I(s)$ will have terms which exponentially increases with time if $\bar{u}_1(s)$ has poles with positive real parts. In order for the trivial steady state (0, 0, 0) to be locally asymptotically stable it is necessary and sufficient that all the poles of $\bar{u}_1(s)$ have negative real parts.

Now Nyquist plot technique will be used which states that ifs is the arc length along a curve circling the right half of the plane, then a curve $\bar{u}_1(s)$ will encircle the origin a number of times equal to the difference between number of poles and number of zeros of $\bar{u}_1(s)$ in the right half of the plane.

Hence the conditions of local asymptotic stability of the trivial steady state (0, 0, 0) is given by:

$$\mathrm{Im}\, F(iy_0) > 0 \tag{11}$$

$$\mathrm{Re}\, F(iy_0) = 0 \tag{12}$$

Where:
F(s) = Given by (10)
iy_0 = The smallest positive root of (12)

Now substituting $s = iy_0$ in F(s) and then separating real and imaginary parts Eq. 12 and 11 becomes respectively

$$\begin{aligned}
&3y_0^2k_1 - k_1(3a^2 + k_1^2) = \left\{3\beta(y_0^2 - k_1^2) - 3a^2\beta + 6abk_1\right\} \\
&\cos y_0\tau - 6y_0(ab - \beta k_1)\sin y_0\tau \\
&M_1\cos 2y_0\tau + M_2y_0\sin 2y_0\tau + M_3\cos 3y_0\tau
\end{aligned} \tag{13}$$

and:

$$\begin{aligned}
&-y_0^3k_1 + 3y_0(a^2 + k_1^2) > \left\{3\beta(y_0^2 - k_1^2) - 3a^2\beta + 6abk_1\right\} \\
&\cos y_0\tau - 6y_0(ab - \beta k_1)\sin y_0\tau \\
&M_1\cos 2y_0\tau + M_2y_0\sin 2y_0\tau + M_3\cos 3y_0\tau
\end{aligned} \tag{14}$$

Where:

$$\begin{aligned}
&M_1 = 3k(\beta^2 + b^2) - 6ab\beta \\
&M_2 = 3(\beta^2 + b^2) \\
&M_3 = -\beta(\beta^2 + 3b^2)
\end{aligned}$$

Now to get the estimate of length of delay we use (13) and (14) which when satisfied simultaneously give the guarantee for local asymptotic stability of trivial steady state (0, 0, 0) of system (2). Our technique will be to find an upper bound y_+ of y_0 independent of τ using (13) and then to estimate τ such that, below that estimated value of τ (14) holds for all values of $y \in [0, y_+]$ and hence in particular $y = y_0$.

From (13):

$$3y_0^2k_1 - k_1(3a^2 + k_1^2) < \left|\begin{matrix}3\beta(y_0^2 - k_1^2) \\ -3a^2\beta + 6abk_1\end{matrix}\right| - 6y_0|ab - \beta k_1|$$
$$|M_1| + |M_2|y_0 + |M_3| \quad (15)$$
$$\Rightarrow 3(k_1 - |\beta|)y_0^2 - (6|ab - \beta k_1| + |M_2|)y_0$$
$$-[k_1^3 + 3|\beta|k_1^3$$
$$+(3a^2 + 6|ab|)k_1 + 3a^2|\beta| + |M_1| + |M_3|] < 0$$

Above relation holds for all values of $y_0 \in [0, y_+)$, where y_+ is the root of:

$$3(k_1 - |\beta|)y_0^2 - (6|ab - \beta k_1| + |M_2|)y_0 - [k_1^3 + 3|\beta|k_1^2$$
$$+(3a^2 + 6|ab|)k_1 + 3a^2|\beta| + |M_1| + |M_3|] = 0 \quad (16)$$

that is $\quad y_+ = \dfrac{B + \sqrt{B^2 + 4AC}}{2A}$

Where:

$$A = 3(k_1 - |\beta|)$$
$$B = (6|ab - \beta k_1| + |M_2|) = 6|ab - \beta k_1| + 3(\beta^2 + b^2)$$
$$C = k_1^3 + 3|\beta|k_1^2 + (3a^2 + 6|ab|)k_1 + 3a^2|\beta| + |M_1| + |M_3| =$$
$$k_1^3 + 3|\beta|k_1^2 + 3\{(|a| + |b|)^2 + \beta^2\}k_1 + |\beta|\{3((a) + |b|)^2 + \beta^2\}$$

At $\tau = 0$ from (14) it can be written as:

$$y_0^2(3k_1 - 3\beta) = k_1(3a^2 + k_1^2) - 3\beta k_1^2 - 3a^2\beta$$
$$+ 6abk_1 + M_1 + M_3 = (k_1 - \beta)^3$$
$$+ 6ab(k_1 - \beta) + 3b^2(k_1 - \beta) \quad (17)$$
$$+ 3a^2(k_1 - \beta)$$
$$\Rightarrow y_{0^2} = \frac{1}{3}(k_1 - \beta)^2 + (a + b)^2$$

At $\tau = 0$ from (15) it can be written as:

$$-y_0^3 + 3y_0(a^2 + k_1^2) > 6y_0(ab - \beta k_1) - M_2y_0$$
$$\Rightarrow y_0^2 < 3(a^2 + k_1^2) - 6(ab - \beta k_1) + 3(\beta^2 + b^2) \quad (18)$$
$$= 3\left\{(k_1 - \beta)^2 + (a + b)^2\right\}$$

As $\frac{1}{3}(k_1 - \beta)^2 + (a + b)^2 < 3\left\{(k_1 - \beta)^2 + (a + b)^2\right\}$, it can be conclude that at $\tau = 0$, (14) is valid at $y = y_0$ given by (17).

So by continuity it will continue to hold for sufficiently small $\tau > 0$ at $y = y_0$.

Now (14) can be written as:

$$y_0^2 + 3(a^2 + k_1^2) < 6(ab - \beta k_1)\cos y_0\tau + M_2\cos 2y_0\tau$$
$$-\frac{\left\{3\beta(y_0^2 - k_1^2) - 3a^2\beta + 6abk_1\right\}}{y_0} \quad (19)$$
$$\sin y_0\tau - \frac{M_1}{y_0}\sin 2y_0\tau - \frac{M_3}{y_0}\sin 3y_0\tau$$

Now for small enough $\tau > 0$ at $y = y_0$, substituting y_0^2 from (13) and (19) we have:

$$[\left\{3\beta(y_0^2 - 5k_1^2 - a^2) - 12abk_1\right\}\cos y_0\tau + \left\{M_1 - 3k_1M_2\right\}$$
$$\cos 2y_0\tau + \left\{M_2y_0 + \frac{3k_1M_1}{y_0}\right\}\sin 2y_0\tau$$
$$+6y_0(ab - \beta k_1) + \frac{3k_1\left\{3\beta(y_0^2 - k_1^2 - a^2) + 6abk_1\right\}}{y_0}$$
$$\sin y_0\tau + M_3\cos 3y_0\tau + \frac{3k_1M_3}{y_0}\sin 3y_0\tau] < k_1(6a^2 + 8k^2) \quad (20)$$
$$\Rightarrow [\left\{3\beta(y_0^2 - 5k_1^2 - a^2) - 12abk_1\right\}(\cos y_0\tau - 1)$$
$$+\left\{M_1 - 3k_1M_2\right\}(\cos 2y_0\tau - 1) + M_3(\cos 3y_0\tau - 1)$$
$$+\left\{6y_0(ab - \beta k_1) + \frac{3k_1\left\{3\beta(y_0^2 - k_1^2 - a^2) + 6abk_1\right\}}{y_0}\right\}$$
$$\sin y_0\tau + \left\{M_2y_0 + \frac{3k_1M_1}{y_0}\right\}\sin 2y_0\tau + \frac{3k_1M_3}{y_0}\sin 3y_0\tau] < \eta$$

Where:

$$\eta = k_1(6a^2 + \beta k_1^2) - 3\beta(y_0^2 - 5k_1^2 - a^2)$$
$$+12abk_1 - M_1 + 3k_1M_2 - M_3 \quad (21)$$

Let us denote LHS of (20) by $\phi(\tau, y)$.

Using the estimate $\sin \tau y < \tau y$ and $1 - \cos 2\tau y = 2\sin^2 \tau y < 2\tau^2 y^2$.

We get $\phi(\tau, y) \le \psi(\tau, y_0)$.

Where:

$$\psi(\tau, y_0) = P\tau^2 + Q\tau \quad (22)$$

P and Q are given by:

$$P = [\frac{1}{2}\left|3\beta(y_0^2 - 5k_1^2 - a^2) + 12abk_1\right|$$
$$+ 2|3k_1M_2 - M_1| + \frac{9}{2}|M_3|]y_0^2 \quad (23)$$
$$Q = [\left|6y_0(ab - \beta k_1) + 3k_1\left\{3\beta(y_0^2 - k_1^2 - a^2) + 6abk_1\right\}\right|$$
$$+ 2\left|M_2y_0^2 - 3K_1M_1\right| + 9k_1|M_3|]$$

It is to be noted that for $0 \le y_0 < y_+$ we have $\phi(\tau,y) \le \psi(\tau,y_0) \le \psi(\tau_+,y_+)$ where τ_+ is the upper limit of τ, below which trivial steady state (0, 0, 0) of system (2) is locally asymptotically stable.

Now if $\psi(\tau_+, y_+) \le \eta$, then $\phi(\tau, y_0) \le \eta$.

Therefore τ_+ is the positive root of $\psi(\tau_+, y_+) = \eta$.

or, τ_+ is the positive root of:

$$P\tau^2 + Q\tau - \eta = 0 \quad (24)$$

where, P, Q are given by (23).

As P, Q both are positive, (24) has at least one negative root, so to make it possible that (24) has a positive root, η should be less than zero. That is τ_+ is given by:

$$\tau_+ = \frac{Q + \sqrt{Q^2 + 4P\eta}}{2P}; \eta < 0 \quad (25)$$

where P, Q are given by (23).

From the discussion above, following theorem can be stated:

Theorem 2: If $k > \gamma + \beta$ and $\eta < 0$ then trivial steady state of system (2) is locally asymptotically stable for every $\tau \in [0, \tau_+)$ where τ_+ is given by (25) P, Q, η are given by (23), $y_0 \in [0, y_+)$ where y_+ is given by (16).

Condition of no stability switching: Now we are to determine the sufficient conditions for which there is no possibility of stability switching for trivial steady state (0, 0, 0) of system (2) in absence of self connection.

Substituting $\beta = 0$ in characteristic Eq. 4, it becomes:

$$\begin{aligned}&\lambda^3 + 3k_1\lambda^2 + 3\lambda[P - A'e^{-\lambda\tau} + C'e^{-2\lambda\tau}]\\&+[Q - R'e^{-2\lambda\tau} + E'e^{-2\lambda\tau}] = 0\end{aligned} \quad (26)$$

Where:

$k_1 = k - \gamma$

$Q = k(k_1^2 + 3a^2)$

$P = (k_1^2 + a^2)$

$R = -6abk_1$

$A' = -2ab$

$E' = 3k_1b^2$

$C' = b^2$

Substituting $\lambda = \mu + i\omega$ in (26) and then separating real and imaginary parts, we get:

$$\begin{aligned}&(\mu + k_1)^3 - 3(\mu + k_1)\omega^2 + 3a^2(\mu + k_1) = 6abe^{-\mu\tau}\\&\{-(\mu + k_1)\cos\omega\tau - \omega\sin\omega\tau\}\\&-3b^2e^{-2\mu\tau}\{(\mu + k_1)\cos 2\omega\tau + \omega\sin\omega\tau\}\end{aligned} \quad (27)$$

and:

$$\begin{aligned}&3\omega[(\mu + k_1)^2 + a^2] - \omega^3 = 6abe^{-\mu\tau}\\&\{-\omega\cos\omega\tau + (\mu + k_1)\omega\sin\omega\tau\}\\&-3b^2e^{-2\mu\tau}\{\omega\cos 2\omega\tau - (\mu + k_1)\sin\omega\tau\}\end{aligned} \quad (28)$$

Let at $\tau = \hat{\tau}, \mu(\hat{\tau}) = 0, \omega(\hat{\tau}) = \hat{\omega}$. That is:

$$\begin{aligned}&k_{1^3} - 3k_1\hat{\omega}^2 + 3a^2k_1 = 6ab[-k_1\cos\{\hat{\omega}\tau - \hat{\omega}\sin\{\hat{\omega}\hat{\tau}]\\&-3b^2[k_1\cos 2\hat{\omega}\tau + \hat{\omega}\sin\{\hat{\omega}\hat{\tau}]\end{aligned} \quad (29)$$

and:

$$\begin{aligned}&3\hat{\omega}[(k_{1^2} + a^{2_1})] - \hat{\omega}^3 = 6ab[\hat{\omega}\cos\{\hat{\omega}\tau + k_1\sin\{\hat{\omega}\hat{\tau}]\\&-3b^2[\hat{\omega}\cos 2\hat{\omega}\tau - k_1\sin\{\hat{\omega}\hat{\tau}]\end{aligned} \quad (30)$$

Squaring and adding both sides of (29) and (30), we have:

$$\begin{aligned}&\hat{\omega}^6 + 3(k_1^2 - 2a^2)\hat{\omega}^4 + 3(k_1^4 + 3a^4)\hat{\omega}^2 + k_1^2(k_1^2 + 3a^2)^2 = 9\\&[4a^2b^2k_1^2 + b^4k_1^2 + 4a^2b^2\hat{\omega}^2 + b^4\hat{\omega}^2 + 4ab^3k_1^2\cos\{\hat{\omega}\\&\hat{\tau} - 4ab^3k_1^2\cos 3\hat{\omega}\hat{\tau} + 4ab^3k_1^2\sin\\&\{\hat{\omega}\hat{\tau} + 8ab^3k_1\hat{\omega}\sin 2\hat{\omega}\hat{\tau} + 4ab^3k_1\hat{\omega}\sin 3\hat{\omega}\hat{\tau}]\end{aligned} \quad (31)$$

Let us denote the right hand side of (31) by $f(\hat{\omega})$. Therefore:

$$\begin{aligned}&f(\hat{\omega}) < 9[4a^2b^2k_1^2 + b^4k_1^2 + 4|ab^3|k_1^2 + 4|ab^3|k_1] + 9\\&[4a^2b^2 + b^4 + 4|ab^3|]\hat{\omega}^2\\&9[8a^2b^2k_1 + 4|ab^3|k_1]\hat{\omega}\end{aligned} \quad (32)$$

$$\begin{aligned}&\Rightarrow \hat{\omega}^6 + 3(k_1^2 - 2a^2)\hat{\omega}^4 + 3(k_1^4 + 3a^2 - 12a^2b^2 - 3b^4\\&-12b^2|ab|)\hat{\omega}^2 - 36b^2|a||k_1|(|b| + 4|a|)\hat{\omega}\\&[k_1^2(k_1^2 + 3a^2)^2 - 36a^2b^2k_1^2 - 9b^4k_1^2 -\\&36b^2k_1^2|ab| - 36b^2|ab||k_1|] < 0\\&\Rightarrow \hat{\omega}^6 + 3(k_1^2 - 2a^2)\hat{\omega}^4 + 3\alpha_1\hat{\omega}^2 - 36\alpha_2\hat{\omega} + \alpha_3 < 0\end{aligned} \quad (33)$$

Where:

$$
\begin{aligned}
&\alpha_1 = (k-\gamma)^4 + 3a^4 - 3b^2(1+|ab|)^2 \\
&\alpha_2 = b^2|a||k-\gamma|(|b|-4|a|) \\
&\alpha_3 = (k-\gamma)^2[\{(k-\gamma)^2 + 3a^2\}^2 - 9b^2 \\
&(2|a|+|b|)^2] - 36b^2|ab||k-\gamma|
\end{aligned}
\tag{34}
$$

A sufficient condition for there to be no stability switches is that the inequality (33) not be satisfied for any real $\hat{\omega}$. This is equivalent to the condition that:

$$\hat{\omega}^6 + 3(k_{12} - 2a^2)\hat{\omega}^4 + 3\alpha_1\hat{\omega}^2 - 36\alpha_2\hat{\omega} + \alpha_3 > 0 \tag{35}$$

for all real $\hat{\omega}$.

Now (36) can be written as:

$$
\begin{aligned}
&\hat{\omega}^6 + 3\left\{(k-\gamma)^2 - 2a^2\right\}\hat{\omega}^4 + 3\alpha_1\left(\hat{\omega} - \frac{6\alpha_2}{\alpha_1}\right)^2 + \\
&\left(\alpha_3 - \frac{108\alpha_2^{\ 2}}{\alpha_1}\right) > 0
\end{aligned}
\tag{36}
$$

for all real $\hat{\omega}$.

From the above discussion it can be stated:

Theorem 3: If:

(i) $(k-\gamma)^2 > 2a^2$
(ii) $\alpha_1 > 0$ and
(iii) $\alpha_1\alpha_2 > 108\ \alpha_2^2$

Then the trivial steady state (0, 0, 0) of system (2) has the same stability character for all values of delay parameter in the absence of self-connection, where α_1, α_2, α_3 are given by (34).

RESULTS

Numerical simulation: To simulate the above described model Numerically, Eq. 2 has been solved by fourth-order Runga-Kutta method prescribing some particular values to synaptic weights α, b, γ, β and decay rate k.

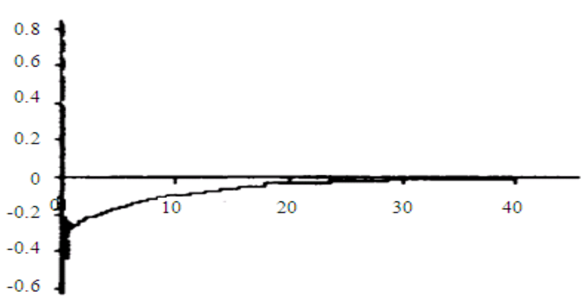

Fig. 1: Here k>+β origin is locally asymptotically stable in absence of delay

In order to do this a program has been developed in Turbo 'C' and then using the results obtained graphs have been plotted in Microsoft Excel and Axum. In Fig. 1-7 time (t) has been taken as independent variable and x (t), y (t), z (t) as dependent variable, that is these give waveform plots. In Fig. 8 corresponding phase portrait of Fig. 7 is drawn.

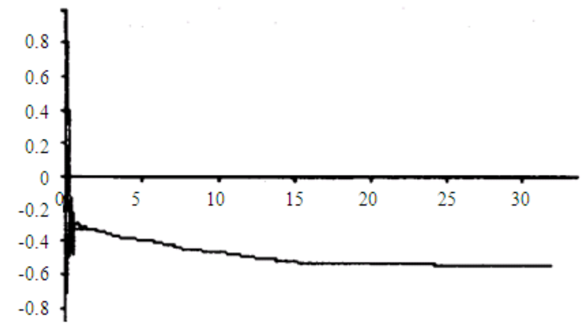

Fig. 2: Here k<γ + β origin is unstable in absence of delay

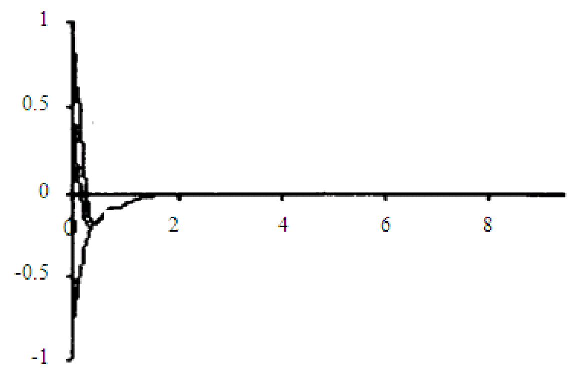

Fig.3: Here k = 1.5, a = 2, b = 0.5, β = 0, γ = -0.7 origin is locally asymptotically stable for τ = 0

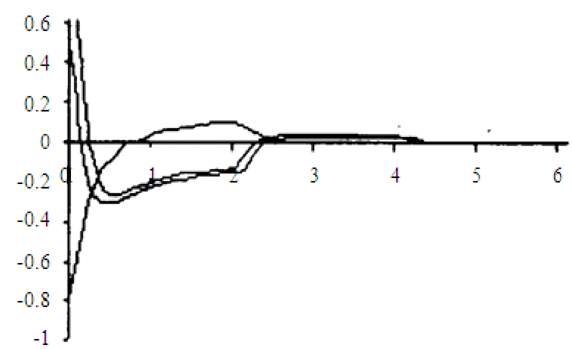

Fig.4: Here k = 1.5, a = 2, b = 0.5, β = 0, γ = -0.7 origin is locally asymptotically stable for τ = 1.8

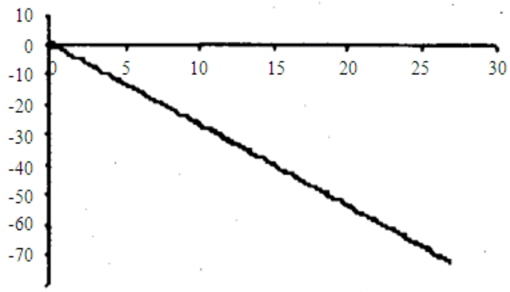

Fig. 5: Here $k = 0.5, a = 2, b = 0.5, \beta = 0, \gamma = 2.7$ origin is unstable stable for $\tau = 0$

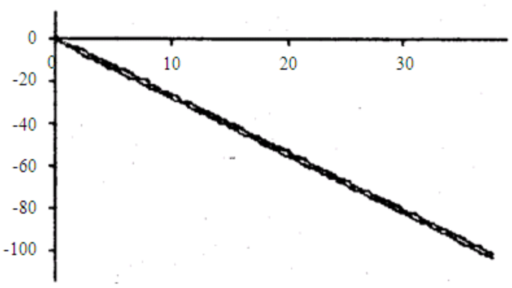

Fig. 6: Here $k = 0.5, a = 2, b = 0.5, \beta = 0, \gamma = 2.7$ origin is unstable for $\tau = 1.8$

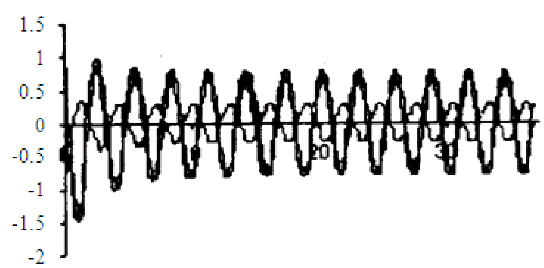

Fig. 7: Here a periodic solution exists about origin

First it has been considered that k = 1, a = 1, y = 0.4, b = 2, β = 0.5, τ = 0 [that is k>γ+β] and in corresponding Fig. 1 it has been shown that origin is locally asymptotically stable.

Now keeping all the values except γ taken in Fig. 1, unchanged only γ has been increased from 0.4-0.6 [that is k<γ+β] and corresponding Fig. 2 show that origin is unstable then.

Then it has been taken k = 1.5, a = 2, γ = -0.7, b = 0.5, β = 0, such that sufficient conditions of no stability switching (stated in Theorem 3) are preserved. For these values it has been shown that origin is stable for τ = 0 and if we increase r arbitrarily (in Fig. 4, τ =1.8), then also origin remains locally asymptotically stable.

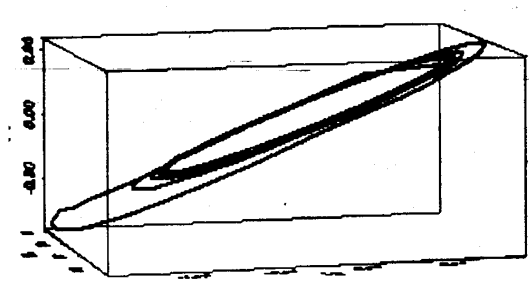

Fig. 8: Phase portrait of Fig. 7

Now taking the values k = 0.5, a = 2, γ = 2.7, b = 0.5, β = 0, Fig. 5 and 6 are drawn for τ = 0 and τ = 1.8 respectively and in both cases origin is unstable.

Now taking the values k = 0.5, a = 2, γ = 2.7, b = 0. 5, β = 0, Fig. 5 and 6 are drawn for τ = 0 and τ =1.2 respectively and in both cases origin is unstable.

That is Theorem 1 is verified by Fig. 1 and 2.

That is Theorem 3 is verified by Fig. 3-6.

At last keeping all the values except τ, taken in Fig. 1, unchanged only τ has been increased to τ = 0.8 and existence of periodic solution for system (2) is confirmed by Fig. 7.

Also from Fig. 2 it is clear that the system has a possibility to have multiple steady states.

DISCUSSION

In this study we analyzed a CNN system composed of three neurons, with discrete delay. To best of our knowledge local stability analysis of such type of cellular neural network model having all the self-connections and inter-connections has not been done. We have derived delay independent sufficient conditions of local asymptotic stability of trivial steady state (0, 0, 0) using Routh-Hurwitz method. In presence of delay, as the characteristic equation (4) is of transcendental type having infinite number of solutions, the local analysis of model (2) is difficult. Here we have used the method of Laplace transformation to determine the delay dependent sufficient conditions of local asymptotic stability. In the delay independent criteria of local asymptotic stability inter-connecting synaptic weights (‘a’, ‘b’) have no role, whereas in delay dependent criteria they have a vital role (η contains parameters ‘a’ and ‘b’). Sufficient conditions of no stability-switching for such a network without self connection has been obtained. From Fig. 7 and 8 it is obvious that, for some specific values of parameters, a periodic solution exists about trivial equilibrium. In our further research work we shall investigate the condition for existence of periodic solutions of this model analytically. From numerical simulation results (shown in Fig. 2), it appears that there may be a possibility of multiple steady states. This part also will be taken in account in our future research.

CONCLUSION

From numerical simulation, it appears that there may be a possibility of multiple steady states of the model. It may be possible to investigate the condition for the existence of periodic solutions of the non-linear model analytically.

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