Topological Beth Model and its Application to Functionals of High Types

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Abstract: Based on the definition of Beth-Kripke model by Dragalin, we describe Beth model from the topological point of view. We show the relation of the topological definition with more traditional relational definition of Beth model that is based on forcing. We apply the topological definition to construct a Beth model for a theory of intuitionistic functionals of high types and to prove its consistency.

Keywords: Intuitionistic Functional, Creating Subject, Topological Model, Beth Model, Forcing Relation.

1. Introduction

The studies of metamathematical properties of non-classical theories are based on a variety of models such as topological models, Beth model and Kripke model. Moschovakis developed a topological model (Moschovakis, 1973) for intuitionistic second-order arithmetic and developed semantics (Moschovakis, 1987) for a theory of lawless sequences. Van Dalen (1978) constructed a Beth model for intuitionistic analysis. In his book (Dragalin, 1987) Dragalin studied a general Beth-Kripke model (BK-model) that combines forcing from Beth model and realizability from Kripke model. The applications of BK-model in (Dragalin, 1987) include different versions of intuitionistic arithmetic and analysis. Most applications of the aforementioned models are for intuitionistic sequences of natural numbers (e.g. choice sequences and lawless sequences). In (Kachapova, 2014; 2015) we created a Beth model for intuitionistic functionals of high types: 1-functional sequences of natural numbers), 2-functional sequences of 1-functionals), ..., (n + 1)-functional sequences of n-functionals. That model was based on the relational definition of Beth model by van Dalen (1978).

In this study we describe the general concept of Beth model from the topological point of view. The topological version has a simpler definition than the relational version and is consistent with the definition of a general algebraic model for an axiomatic theory as in (Dragalin, 1987). In this study we apply the topological version of Beth model to the intuitionistic theory SLP of high-order functionals from (Kachapova, 2015), including lawless functionals and the "creating subject". It can be seen that the topological version of Beth model simplifies some constructions and consistency proofs. In metamathematical proofs we use classical logic.

2. Preliminary Concepts

The introductory theory in this section follows the book by Dragalin (1987).

2.1. Logical-Mathematical Language. Axiomatic Theory

Definition 2.1

A logical-mathematical language of first order (or logical-mathematical language, or just language in short) is defined as a sequence \( \Omega = \langle \text{Srt}, \text{Cnst}, \text{Fn}, \text{Pr} \rangle \), where

1) \text{Srt} is a non-empty set of sorts of objects and for each sort \( \pi \in \text{Srt} \) there is a countable collection of variables of this sort;
2) \text{Cnst} is the set of all constants of the language;
3) \text{Fn} is the set of all functional symbols of the language;
4) \text{Pr} is the set of all predicate symbols of the language.

In the language \( \Omega \) we can construct terms, atomic formulas and formulas as usual.

Definition 2.2

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3) \text{Fn} is the set of all functional symbols of the language;
4) \text{Pr} is the set of all predicate symbols of the language.

Axiomatic theory (or just theory in short) is defined as \( \text{Th} = \langle \Omega, l, A \rangle \), where each of the three objects is described as follows.

1) \( \Omega \) is a logical-mathematical language.
2) \( l \) is the logic of the theory. We will use only the intuitionistic logic HPC (Heyting’s predicate calculus).
3)  $A$ is some set of closed formulas (i.e., formulas without parameters) of the language $\Omega$; it is called the set of non-logical axioms of $Th$. When axioms are stated as non-closed formulas, it means that they must be closed by universal quantifiers over all parameters.

The notation $Th \vDash \varphi$ (formula $\varphi$ is derivable in the theory $Th$) means that $\varphi$ is derivable in the logic $l$ from a finite subset of the axiom set $A$.

2.2. Pseudo Boolean Algebras

Definition 2.3

A pair $\langle B, \leq \rangle$ is called a pseudo Boolean algebra (p.B.a.) if $B$ is a set, $\leq$ is a binary relation on $B$ and they satisfy the following 9 conditions.

1)  $a \leq a$.
2)  $a \leq b \& b \leq c \Rightarrow a \leq c$.
   
   For any $a, b \in B$ there exists an element $a \land b \in B$ such that:
3)  $a \land b \leq a, a \land b \leq b$.
4)  $c \land a \leq c \land b$.
   
   For any $a, b \in B$ there exists an element $a \lor b \in B$ such that:
5)  $a \leq a \lor b, b \leq a \lor b$.
6)  $a \land c \leq a \land b, c \land b$.
   
   For any $a, b \in B$ there exists an element $(a \lor b) \in B$ such that:
7)  $a \land (a \lor b) \leq b$.
8)  $c \land a \leq b \Rightarrow c \leq (a \lor b)$.
9)  There exists an element $\bot \in B$ such that for any $a \in B$: $\bot \leq a$.

The element $\bot$ is "falsity" and the element $T = (\bot \lor \bot)$ is "truth".

Definition 2.4

Suppose $\langle B, \leq \rangle$ is a p.B.a., $W \subseteq B$ and $a \in B$.

1)  $a$ is denoted $\land W$ and is called the intersection or conjunction of $W$ if
   a)  $(\forall c \in W) (a \leq c)$ and
   b)  for any $d \in B$, $(\forall c \in W) (d \leq c) \Rightarrow d \leq a$.
2)  $a$ is denoted $\lor W$ and is called the union or disjunction of $W$ if
   a)  $(\forall c \in W) (c \leq a)$ and
   b)  for any $d \in B$, $(\forall c \in W) (c \leq d) \Rightarrow a \leq d$.

Definition 2.5

A p.B.a. $\langle B, \leq \rangle$ is called complete if for any $W \subseteq B$ there exist $\land W \in B$ and $\lor W \in B$.

2.3. Algebraic Model of a Language

The following is definition of an algebraic model with constant domains. For brevity we call it just an algebraic model.

Definition 2.6

An algebraic model of the language $\Omega$ is a sequence $A = \langle B, D, \text{Cnst}, \hat{F}_n, \hat{Pr} \rangle$ of objects defined as follows.

1)  $B$ is a complete p.B.a.
2)  To each sort $\pi$ of the language $\Omega$ the function $D$ assigns a non-empty set $D_\pi$, which is called the domain of sort $\pi$.
3)  To each constant of sort $\pi$ the function $\text{Cnst}$ assigns an element $c \in D_\pi$.
4)  Function $\hat{F}_n$ assigns values to functional symbols of $\Omega$ in the following way.
   Let $f$ be a functional symbol of sort $\pi$ with arguments of sorts $\pi_1, \ldots, \pi_k$. Then for any $q_1 \in D_{\pi_1}, \ldots, q_k \in D_{\pi_k}$, $q \in D_\pi$, $\hat{F}_n(f, q_1, \ldots, q_k, q) \in B$.
   $\hat{F}_n$ satisfies the following conditions (1) and (2).

$$\lor \left[ \hat{F}_n(f, q_1, \ldots, q_k, q) \mid q \in D_\pi \right] = T.$$  \hspace{1cm} (1)

If $q \neq q'$, then:

$$\hat{F}_n(f, q_1, \ldots, q_k, q) \land \hat{F}_n(f, q_1, \ldots, q_k, q') = \bot.$$  \hspace{1cm} (2)

5)  If $P$ is a predicate symbol of $\Omega$ with arguments of sorts $\pi_1, \ldots, \pi_\ell$, then for any $q_1 \in D_{\pi_1}, \ldots, q_\ell \in D_{\pi_\ell}$:

$$\hat{Pr}(P, q_1, \ldots, q_\ell) \in B.$$  \hspace{1cm} (3)

An evaluated term is a term of the language $\Omega$, in which all parameters are replaced by elements from suitable domains. An evaluated formula is a formula of the language $\Omega$, in which all parameters are replaced by elements from suitable domains.

Definition 2.7

Suppose $t$ is an evaluated term of sort $\pi$ and $q \in D_\pi$. The set $\| t \sim q \|$ is defined by induction on the complexity of $t$. 

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1) If \( t \) is an element of \( D_\pi \), then
\[
\| t \sim q \| = \begin{cases} \top & \text{if } t = q, \\ \bot & \text{otherwise}. \end{cases}
\]

2) If \( t \) is a constant \( c \) and \( \pi = \widehat{\text{Cast}}(c) \), then
\[
\| t \sim q \| = \begin{cases} \top & \text{if } \pi = q, \\ \bot & \text{otherwise}. \end{cases}
\]

3) Suppose \( t = f(t_1, \ldots, t_k) \), where each \( t_i \) is an evaluated term of sort \( \pi_i, i = 1, \ldots, k \). Then
\[
\| t \sim q \| = \bigvee \left\{ \widehat{f} \left( t_1, \ldots, t_k, q_1, \ldots, q_k \right) \wedge t_i \sim q_i \ \bigwedge \ldots \wedge t_k \sim q_k \ : q_i \in D_{\pi_i}, \ldots, q_k \in D_{\pi_k} \right\}.
\]

**Lemma 2.8**

For an evaluated term \( t \) of sort \( \pi \):
\[
\bigvee \left\{ \| t \sim q \| : q \in D_\pi \right\} = \top.
\]

**Definition 2.9**

For an evaluated formula \( \varphi \), \( \| \varphi \| \) is defined by induction on the complexity of \( \varphi \).

1) Suppose \( \varphi \) is an atomic formula \( P(t_1, \ldots, t_k) \), where each \( t_i \) is an evaluated term of sort \( \pi_i, i = 1, \ldots, k \). Then
\[
\| \varphi \| = \bigvee \left\{ \widehat{P} \left( t_1, \ldots, t_k, q_1, \ldots, q_k \right) \wedge t_i \sim q_i \ \bigwedge \ldots \wedge t_k \sim q_k \ : q_i \in D_{\pi_i}, \ldots, q_k \in D_{\pi_k} \right\}.
\]

In particular, if each \( t_i \) is an element \( p_i \in D_{\pi_i} \), then
\[
\| \varphi \| = \widehat{P} \left( p_1, \ldots, p_k \right).
\]

2) \( \| \psi \wedge \eta \| = \| \psi \| \wedge \| \eta \| \).
3) \( \| \psi \vee \eta \| = \| \psi \| \vee \| \eta \| \).
4) \( \| \psi \supset \eta \| = \| \psi \| \supset \| \eta \| \).
5) \( \bot = \bot \).
6) \( \| \forall x \varphi(x) \| = \bigwedge \{ \| \varphi(q) \| : q \in D_\pi \} \), where \( x \) is a variable of sort \( \pi \).
7) \( \| \exists x \varphi(x) \| = \bigvee \{ \| \varphi(q) \| : q \in D_\pi \} \), where \( x \) is a variable of sort \( \pi \).

**Theorem 2.10. Soundness Theorem.**

Suppose \( \varphi \) is a formula of \( \mathcal{Q} \) and \( \varphi' \) is \( \varphi \) evaluated by elements of appropriate domains. Then the following hold.

1) \( HPC \vdash \varphi \Rightarrow \| \varphi' \| = \top \).
2) If \( \varphi \) is a closed formula, then
\[
HPC \vdash \varphi \Rightarrow \| \varphi \| = \top.
\]

2.4. Topological Space

**Definition 2.11**

A pair \( Y = (X, S) \) is called a topological space if it satisfies the following conditions:

1) \( S \) is a collection of subsets of \( X \).
2) \( \emptyset \in S, X \in S \).
3) \( A, B \in S \Rightarrow A \cap B \in S \).
4) \( W \subseteq S \Rightarrow \bigcup W \in S \).

Elements of \( S \) are called open sets and \( S \) is called the topology on \( X \).

**Definition 2.12**

Suppose \( Y = (X, S) \) is a topological space. A collection \( H \) of subsets of \( X \) is called a base of the topology \( S \) if every \( A \in S \) can be written as a union of elements of \( H \).

Then we say that \( H \) generates the topology \( S \).

**Lemma 2.13**

Suppose \( H \) is a collection of subsets of \( X \). \( H \) is a base of some topology on \( X \) if it satisfies the following two conditions:

1) \( X = \bigcup H \),
2) \( (\forall A_1, A_2 \in H) ((\forall x \in A_1 \cap A_2) (\exists A_3 \in H) (x \in A_3 \subseteq A_1 \cap A_2)) \).

**Lemma 2.14**

Suppose \( Y = (X, S) \) is a topological space. Then \( (S, \subseteq) \) is a complete p.B.a. with the operations given by:
\[
\bot = \emptyset, \quad T = X, \quad A \wedge B = A \cap B, \quad A \vee B = A \cup B, \quad A \supset B = \{ x \in X \mid (\exists C \in S) (x \in C \cap A \subseteq B) \}, \quad \wedge W = \bigcap W, \quad \vee W = \bigcup W.
\]

2.5. Completion Operator

**Definition 2.15**

Suppose \( Y = (B, \subseteq) \) is a p.B.a. and \( \varepsilon : B \to B \).

1) \( \varepsilon \) is called a completion operator on \( Y \) if the following 4 conditions are satisfied for any \( a, b \in B \):
a) \( a \leq c a, \)

b) \( a \leq b \Rightarrow c a \leq c b, \)

c) \( c c a = c a, \)

d) \( a = c a \land a \leq c b \Rightarrow a \leq c (a \land b). \)

2) An element \( a \in B \) is called complete if \( c a = a. \)

3) Denote \( C(B) \) the set of all complete elements of \( B \).

**Lemma 2.16**

Suppose \( Y = \langle B, \leq \rangle \) is a p.B.a. and \( c \) is a completion operator on \( Y \). Denote \( Y^c = \langle c(B), \leq \rangle \). Then \( Y^c \) is also a p.B.a. with the operations given by:

- \( \bot^c = c(\bot) \),
- \( a \land^c b = a \land b \),
- \( a \lor^c b = c(a \lor b) \),
- \( a \Rightarrow^c b = a \Rightarrow b \).

If \( Y \) is a complete p.B.a., then \( Y^c \) is also a complete p.B.a. and

\[
\land^c + W = \land W, \\
\lor^c W = c (\lor W).
\]

3. Beth Model

The following definitions of Beth frame, Beth algebra and Beth model are modified from the definitions of Beth-Kripke frame, Beth-Kripke algebra and Beth-Kripke model given in the book (Dragalin, 1987). Some accompanying lemmas and theorems are proven here; other proofs can be found in (Dragalin, 1987).

### 3.1. Beth Frame

A **tree** is a set \( M \) with partial order \( \leq \) such that for any \( x \in M \) the set \( \{ y \in M \mid y > x \} \) is a well-ordered set. We fix a tree \( \langle M, \leq \rangle \) till the end of this section.

**Definition 3.1**

1) A **path** in \( M \) is a maximal linearly ordered subset of \( M \).
2) A **path through an element** \( x \) is a path containing \( x \).
3) For any \( a \in M \) denote

\[
\mathcal{Q}(a) = \{ \text{all paths through } a \}.
\]

The triple \( \langle M, \leq, \mathcal{Q} \rangle \) is called a **Beth frame**.

### 3.2. Beth Algebra

**Definition 3.2**

1) For any \( x \in M, [x] = \{ y \in M \mid y \leq x \} \) is called the **peaked cone** generated by \( x \)

2) Collection of all sets \( [x], x \in M, \) satisfies the conditions of Lemma 2.13, therefore it is a base of a topology on \( M \). This topology is called the **order topology**

3) Thus, we get a topological space \( \langle M, Op(M) \rangle \), where \( Op(M) \) denotes the set of all open subsets of \( M \) in the order topology

By Lemma 2.14, \( \langle Op(M), \subseteq \rangle \) is a complete p.B.a.

**Lemma 3.3**

For any subset \( A \) of \( M \):

\( A \) is open in the order topology

\[
\iff (\forall a \in A)(\forall b \in M)(b \leq a \Rightarrow b \in A).
\]

**Proof**

Follows from the definition. \( \Box \)

**Definition 3.4**

For any \( U \subseteq M \) denote:

\[
CU = \{ a \in M \mid (\forall S \in \mathcal{Q}(a))(S \cap U \neq \emptyset) \}.
\]

**Lemma 3.5**

The operator \( C \) defined above has the following properties.

1) \( C : Op(M) \rightarrow Op(M) \)
2) \( C \) is a completion operator on the p.B.a. \( \langle Op(M), \subseteq \rangle \).
3) For any open subsets \( A \) and \( B \) of the set \( M \):

\[
CA \cap CB = C(A \cap B).
\]

**Definition 3.6**

Denote \( Y = \langle Op(M), \subseteq \rangle \). It is a complete p.B.a. The operator \( C \) from Definition 3.4 is a completion operator. Thus, by Lemma 2.16, \( Y^c = \langle c(Op(M)), \subseteq \rangle \) is a complete p.B.a. It is denoted \( B(M, \leq, \mathcal{Q}) \) and is called the **Beth algebra** generated by the Beth frame \( \langle M, \leq, \mathcal{Q} \rangle \).

**Theorem 3.7. Operations in Beth Algebra**.

Operations in the Beth algebra are given by:

1) \( U \land V = U \cap V \)
2) \( U \lor V = C(U \cup V) \)
3) \( U \supseteq V = \{ a \in M \mid (\forall b \leq a)(b \in U \Rightarrow b \in V) \} \)
4) \( T = M \)
5) \( \bot = \{ a \in M \mid \mathcal{Q}(a) = \emptyset \} \)
6) \( \land W = \cap W \)
7) \( \lor W = C(UW) \).
Proof
Follows from Lemma 2.16. □

Lemma 3.8
1) For any subsets A and B of the set M:
   \[ CA \cup CB = C(A \cup B). \]
2) For any collection W of subsets of M:
   \[ \bigvee \{CA \mid A \in W\} = C(\bigcup W). \]

Proof
1) We need to prove:
   \[ C(CA \cup CB) = C(A \cup B). \]

   By properties of completion operator, \( A \subseteq CA, A \cup B \subseteq CA \cup CB \) and \( C(A \cup B) \subseteq C(CA \cup CB). \)
   To prove the inverse, it is sufficient to show that \( CA \cup CB \subseteq C(A \cup B). \)
   Since \( A \subseteq A \cup B \), we have \( CA \subseteq C(A \cup B). \)
   Similarly \( CB \subseteq C(A \cup B) \) and \( CA \cup CB \subseteq C(A \cup B). \)
2) is proven similarly. □

3.3. Beth Model

Beth model is a particular case of an algebraic model.

We will use the notation \( f: a \rightarrow b \), which means that \( f \) is a partial function from set \( a \) to set \( b \). The notation \( Z \downarrow \) means that the object \( Z \) is defined.

Definition 3.9

A Beth model for a language \( \Omega = \langle Srt, Cnst, Fn, Pr \rangle \) is an algebraic model \( \{B, D, Cnst, Fn, Pr\} \) with specific definitions of \( B, \widehat{Fn} \) and \( Pr \).

1) \( B = B(M, \leq, Q) \) is the Beth algebra generated by the Beth frame \( \langle M, \leq, Q \rangle \).
   Elements of \( M \) are denoted \( \alpha, \beta, \gamma, \ldots \).

2) Before defining \( \widehat{Fn} \) we define a function \( \overline{Fn} \).
   To each \( \alpha \in M \) and each functional symbol \( f(x_1, \ldots, x_k) \) with sort \( \pi \) and arguments of sorts \( \pi_1, \ldots \), \( \pi_k \), respectively, the function \( \overline{Fn} \) assigns a partial function \( f^{[\alpha]}: D_{\pi_1} \times \cdots \times D_{\pi_k} \rightarrow D_{\pi} \) such that for any \( q_1 \in D_{\pi_1}, \ldots, q_k \in D_{\pi_k}, q \in D_{\pi} \):
   \[ \{ \alpha \mid f^{[\alpha]}(q_1, \ldots, q_k) = q \} \]
   is an open set and
   \[ C(\{ \alpha \mid f^{[\alpha]}(q_1, \ldots, q_k) \downarrow \}) = M. \]

   where \( C \) is from Definition 3.4.
   \( \widehat{Fn} \) is defined by:
   \[ \widehat{Fn}(f, q_1, \ldots, q_k, q) = C(\{ \alpha \mid f^{[\alpha]}(q_1, \ldots, q_k) = q \}). \]

   Clearly, \( \widehat{Fn} \) satisfies the conditions (1) and (2) of the Definition 2.6 of an algebraic model.

3) Before defining \( \widehat{Pr} \) we define a function \( \overline{Pr} \).

   To each predicate symbol \( P(x_1, \ldots, x_k) \) with arguments of sorts \( \pi_1, \ldots, \pi_k \) respectively and \( a \in D_{\pi_1}, a \in D_{\pi_k} \) the function \( \overline{Pr} \) assigns an open set \( \overline{Pr}(P, q_1, \ldots, q_k) \).
   \( \widehat{Pr} \) is defined by:
   \[ \widehat{Pr}(P, q_1, \ldots, q_k) = C(\overline{Pr}(P, q_1, \ldots, q_k)). \]

   Clearly, \( \widehat{Pr} \) satisfies the condition (3) of the Definition 2.6 of an algebraic model.

   This completes the definition of Beth model.

Notes

1) The component \( B \) of the Beth model is uniquely defined by the tree \( \langle M, \leq \rangle \).
2) The component \( \widehat{Fn} \) is uniquely defined by the function \( \overline{Fn} \).
3) The component \( \widehat{Pr} \) is uniquely defined by the function \( \overline{Pr} \).
4) Thus, to construct a Beth model, one needs to specify a tree \( \langle M, \leq \rangle \), a domain function \( D \) and functions \( Cnst, \widehat{Fn} \) and \( \widehat{Pr} \).

Definition 3.10

For any \( \alpha \) and evaluated term \( t \) of sort \( \pi \) we define \( t^{[\alpha]} \) by induction on the complexity of \( t \).

1) If \( t \) is an element of \( D_{\pi} \), then \( t^{[\alpha]} = t \).
2) If \( t \) is a constant \( c \), then \( t^{[\alpha]} = \overline{Cnst}(\alpha) = c \).
3) If \( t = f(t_1, \ldots, t_k) \), then \( t^{[\alpha]} = f^{[\alpha]}(t_1^{[\alpha]}, \ldots, t_k^{[\alpha]}). \)

   Since \( f \) is a partial function, \( t^{[\alpha]} \) is not always defined.
The set \( || \sim q || \) was defined in Definition 2.7 for a general algebraic model. Next lemma specifies it for the Beth model.

**Lemma 3.11**

Suppose \( t \) is an evaluated term of sort \( \pi \) and \( q \in D_\pi \).

1) \( \{ \alpha \mid t^{\alpha} = q \} \) is an open set;

2) \( || t \sim q || = C(\{ \alpha \mid t^{\alpha} = q \}) \).

**Proof**

1) Proof is by induction on the complexity of \( t \).

If \( t \) is an element of \( D_\pi \), then \( \{ \alpha \mid t^{\alpha} = q \} \) is either \( T \) or \( \bot \); in both cases it is an open set.

When \( t \) is a constant \( c \), the proof is similar.

Suppose \( t = f(t_1, \ldots, t_k) \), where each \( t_i \) is an evaluated term of sort \( \pi_i, i = 1, \ldots, k \).

Then \( f^{(s)}(t_1^{\alpha}, \ldots, t_k^{\alpha}) = q \) implies that each \( t_i^{\alpha} \) is defined.

So

\[
\{ \alpha \mid t^{\alpha} = q \} = \bigcup_{q \in D_\pi} \left( \{ \alpha \mid f^{(s)}(q_1, \ldots, q_k) = q \} \cap \bigwedge_{t_i^{\alpha} = q} \{ \alpha \mid t_i^{\alpha} = q \} \right).
\]

By condition (4) of the definition of Beth model, \( \{ \alpha \mid f^{(s)}(q_1, \ldots, q_k) = q \} \) is an open set. Each \( \{ \alpha \mid t_i^{\alpha} = q \} \) is an open set by the inductive assumption.

Therefore \( \{ \alpha \mid t^{\alpha} = q \} \) is an open set.

2) Proof is by induction on the complexity of \( t \).

If \( t \) is an element of \( D_\pi \), then \( t^{\alpha} = t \). So each side of the equality is \( T \) if \( t = q \), \( \bot \) otherwise.

If \( t \) is a constant \( c \), then \( t^{\alpha} = c \). So each side of the equality is \( T \) if \( \alpha = q \), \( \bot \) otherwise.

Suppose \( t = f(t_1, \ldots, t_k) \), where each \( t_i \) is an evaluated term of sort \( \pi_i, i = 1, \ldots, k \).

Then

\[
\{ \alpha \mid t^{\alpha} = q \} = \bigcup_{q \in D_\pi} \left( \{ \alpha \mid t^{\alpha} = q \} \cap \bigwedge_{t_i^{\alpha} = q} \{ \alpha \mid t_i^{\alpha} = q \} \right).
\]

By the inductive assumption each \( \{ \alpha \mid t_i^{\alpha} = q \} \) is an open set. Then by Lemma 3.5.3):

\[
\widehat{F_n}(f, q_1, \ldots, q_k) \land || t \sim q || = C(\{ \alpha \mid t^{\alpha} = q \}) = C\left(\bigwedge_{t_i^{\alpha} = q} \{ \alpha \mid t_i^{\alpha} = q \}\right).
\]

So

\[
|| t \sim q || = \bigvee \{ \widehat{F_n}(f, q_1, \ldots, q_k) \land || t_i \sim q_i || \mid q_i \in D_\pi, i = 1, \ldots, k \}.
\]

\[
\bigvee \{ \{ \alpha \mid t_i^{\alpha} = q_i \} \cap \bigwedge_{t_i^{\alpha} = q_i} \{ \alpha \mid t_i^{\alpha} = q_i \} \mid q_i \in D_\pi, i = 1, \ldots, k \}
\]

by Lemma 3.8.2) and (6). □

**Lemma 3.12**

For an evaluated term \( t \) of sort \( \pi \):

\[
C\left(\bigcup_{q \in D_\pi} \{ \alpha \mid t^{\alpha} = q \}\right) = T.
\]

**Proof**

By Lemma 2.8, \( \forall \{ || t \sim q || \mid q \in D_\pi \} = T \). So by Lemma 3.11.2) and Lemma 3.8.2),

\[
T = \bigvee \{ \{ \alpha \mid t_i^{\alpha} = q_i \} \mid q_i \in D_\pi \}
\]

by Lemma 3.8.2) and (6). □
Lemma 3.13
Suppose \( P(t_1, ..., t_k) \) is an atomic formula, where each 
\( t_i \) is an evaluated term of sort \( \pi_i, i = 1, ..., k \). 
Then:

\[
P\left(t_1, ..., t_k\right) = \bigcup \{ \text{Pr}(P, q_1, ..., q_k) \}
\]

\[
\cap \{ \alpha \mid t_1^{\alpha} = q_1, \& \ldots \& t_k^{\alpha} = q_k \}.
\]

Proof
By Lemma 3.11.2):

\[
\exists \text{Pr}(P, q_1, ..., q_k) \cap \{ \alpha \mid t_1^{\alpha} = q_1 \} \cap \ldots
\]

\[
\cap \{ \alpha \mid t_1^{\alpha} = q_1, \& \ldots \& t_k^{\alpha} = q_k \}.
\]

So
\[
\exists \text{Pr}(P, q_1, ..., q_k) \cap \{ \alpha \mid t_1^{\alpha} = q_1, \& \ldots \& t_k^{\alpha} = q_k \}.
\]

by Lemma 3.8.2). □

Definition 3.14
A forcing relation \( \mathrel{\Vdash} \) is defined by:
for any \( \alpha \) and evaluated formula \( \varphi \)

\[
\alpha \mathrel{\Vdash} \varphi \iff \alpha \in [\varphi].
\]

Theorem 3.15. Properties of the Forcing Relation.
1) Monotonicity of forcing:
\( \alpha \mathrel{\Vdash} \varphi \wedge \beta \leq \alpha \Rightarrow \beta \mathrel{\Vdash} \varphi \).
2) \( \alpha \mathrel{\Vdash} \varphi \Rightarrow (\forall S \in Q(\alpha)) (\exists \beta \in S)(\beta \mathrel{\Vdash} \varphi). \)
3) Suppose \( \varphi \) is an atomic formula \( P(t_1, ..., t_k) \), where 
each \( t_i \) is an evaluated term of sort \( \pi_i, i = 1, ..., k \). Then

\[
\alpha \mathrel{\Vdash} \varphi \iff (\forall S \in Q(\alpha))(\exists \beta \in S)(\exists t_i \in D_n)...
\]

\[
(3q_i \in D_n) \left[ t_i^{\beta} = q_i \& \ldots \& t_k^{\alpha} = q_k \& \beta \mathrel{\Vdash} \text{Pr}(P, q_1, ..., q_k). \right]
\]

4) \( \alpha \mathrel{\Vdash} \varphi \wedge \psi \iff (\alpha \mathrel{\Vdash} \varphi) \& (\alpha \mathrel{\Vdash} \psi) \)
5) \( \alpha \mathrel{\Vdash} \varphi \vee \psi \iff (\forall S \in Q(\alpha))(\exists \beta \in S)((\beta \mathrel{\Vdash} \varphi) \or (\beta \mathrel{\Vdash} \psi)) \)
6) \( \alpha \mathrel{\Vdash} \varphi \Rightarrow \psi \iff (\forall \beta \leq \alpha) (\beta \mathrel{\Vdash} \varphi \Rightarrow \beta \mathrel{\Vdash} \psi) \)
7) \( \alpha \mathrel{\Vdash} \bot \iff \bot \)
8) \( \alpha \mathrel{\Vdash} \neg \varphi \iff (\forall \beta \leq \alpha) \neg (\beta \mathrel{\Vdash} \varphi) \)
9) \( \alpha \mathrel{\Vdash} \forall x \varphi(x) \iff (\forall \beta \leq \alpha) (\forall q \in D_n) (\beta \mathrel{\Vdash} \varphi(q), \) where \( x \) is a variable of sort \( \pi \)
10) \( \alpha \mathrel{\Vdash} \exists x \varphi(x) \iff (\forall S \in Q(\alpha))(\exists \beta \in S)(\exists q \in D_n) (\beta \mathrel{\Vdash} \varphi(q)), \) where \( x \) is a variable of sort \( \pi \)

Proof
In the Beth model \( [\varphi] \in C(Op(M)), \) so \( [\varphi] \) is both an open set and a complete element.
1) Follows from the fact that \( [\varphi] \) is an open set
2) Follows from the fact that \( [\varphi] \) is a complete element
3) Follows from Lemma 3.13

The rest follow from Definition 2.9 and Theorem 3.7. □

4. Axiomatic Theory of Functionals of High Types
As an application of the topological definition of Beth model, we construct a Beth model for an intuitionistic theory \( SLP \) of functionals of high types. We introduced this theory in (Kachapova, 2015) and constructed its Beth model using the van Daalen’s relational definition (Van Dalen, 1978). The topological definition outlined here simplifies some parts of the construction and proofs.

We define the theory \( SLP \) in three steps: We introduce axiomatic theories \( L, LP \) and \( SLP \). In these theories variables have superscripts for types. A superscript for a variable is usually omitted when the variable is used for the second time or more in a formula (so its type is clear).

4.1. Axiomatic Theory \( L \)
The language of theory \( L \) has the following variables:
\( x, y, z, ... \) over natural numbers (variables of type 0) and
variables of type \( n (n \geq 1) \):

\( F^n, G^n, H^n, ... \) over \( n \)-functionals (functionals of type \( n \));
\( A^n, B^n, C^n, ... \) over lawlike \( n \)-functionals;
\( \beta^n, \gamma^n, \alpha^n, ... \) over lawless \( n \)-functionals.
Constants: 0 of type 0 and for each \( n \geq 1 \) a constant \( K^n \) (an analog of 0 for type \( n \)).

Functional symbols: \( N^n, Ap^n \) (\( n \geq 1 \)), \( S \) for successor, and +.

Predicate symbols: \( \equiv_n \) for each \( n \geq 0 \).

Terms and \( n \)-functionals are defined recursively as follows.

1. Every numerical variable is a term
2. Constant 0 is a term
3. Every variable of type \( n \) is an \( n \)-functional
4. Constant \( K^n \) is an \( n \)-functional
5. If \( t \) is a term, then \( St \) is a term
6. If \( t_1 \) and \( t_2 \) are terms, then \( t_1 + t_2 \) and \( t_1 t_2 \) are terms
7. If \( Z \) is an \( n \)-functional, then \( N^n(Z) \) is an \( n \)-functional
8. If \( Z \) is a 1-functional, \( t \) is a term, then \( Ap^1(Z, t) \) is a term
9. If \( Z \) is an \((n + 1)\)-functional, \( t \) is a term, then \( Ap^{n+1}(Z, t) \) is an \( n \)-functional

\( Ap^n(Z, t) \) is interpreted as the result of application of functional \( Z \) to term \( t \). We also denote \( Ap^n(Z, t) \) as \( Z(t) \).

Here 1-functional is interpreted as a function from natural numbers to natural numbers and \((n + 1)\)-functional is interpreted as a function from natural numbers to \( n \)-functionals.

Atomic formulas:

\( t =_n t \), where \( t \) and \( r \) are terms;

\( Z =_n V \), where \( Z \) and \( V \) are \( n \)-functionals (\( n \geq 1 \)).

Formulas are constructed from atomic formulas using logical connectives and quantifiers. For a formula \( \varphi \) its sort \( \text{sort}(\varphi) \) is the maximal type of parameters in \( \varphi \); it is 0 if \( \varphi \) has no parameters.

The theory \( L \) has intuitionistic predicate logic \( HPC \) with equality axioms and the following non-logical axioms.

1. \( Sx \neq 0, Sx = Sy \rightarrow x = y \).
2. \( x + 0 = x, x + Sy = S(x + y) \).
3. \( x0 = 0, xSy = xy + x \).
4. Induction for natural numbers:
\[ \varphi(0) \land \forall x (\varphi(x) \supset \varphi(Sx)) \supset \forall x \varphi(x), \]

where \( \varphi \) is any formula of \( L \).

The axioms 1 - 4 define arithmetic at the bottom level.

5. \( K^{n+1}(x) = K^n, (\neg(N^n(F^n) = K^n)) \)

6. \( N^{n+1}(F^n+1)(x) = N^n(F^n(x)), N^n(F^n) = N^n(G^n) \supset F^n = G^n \)

The axioms 5 and 6 describe \( K^n \) and \( N^n \) as analogs of zero and successor function, respectively, on level \( n \).

7. Principle of primitive recursive completeness of lawlike functions:
\[ \exists t \forall x (A(x) = t), \]

where \( t \) is any term containing only variables of type 0 and variables over lawlike 1-functionals.

Denote \( L_s \), the fragment of the theory \( L \) with types not greater than \( s \). The language of \( L_s \) has one type of functionals and is essentially the language of the intuitionistic analysis \( FIM \).

4.2. Axiomatic Theory \( LP \)

This theory is obtained from \( L \) by adding predicate symbols and axioms for the "creating subject".

Gödel numbering of symbols and expressions can be defined for the language of \( L \). For an expression \( q \) we denote \( \vdash q \) the Gödel number of \( q \) in this numbering.

The language of theory \( LP \) is the language of \( L \) with an extra predicate symbol \( Pr_{\tau, s}(z, \bar{X}) \) for every formula \( \varphi \) of \( L \), which has all its parameters in the list \( \bar{X} \); here \( \bar{X} \) is a list of variables \( X_1, ..., X_s \). Traditionally this symbol is denoted \( \vdash_{\bar{X}} \varphi \ (\bar{X}) \); it means that the formula \( \varphi \ (\bar{X}) \) has been proven by the "creating subject" at time \( z \).

Axioms of the theory \( LP \) are all axioms of \( L_s \), where the axiom schemata are taken for the formulas of the new language, and the following three axioms; in all of them \( \varphi \) is an arbitrary formula of \( L \).

Axioms for the "creating subject":

\[ (CS1) \quad (\vdash_{\bar{X}} \varphi) \lor (\vdash_{\bar{X}} \neg \varphi); \]

\[ (CS2) \quad (\vdash_{\bar{X}} \varphi) \rightarrow (\vdash_{\bar{Y}} \varphi); \]

\[ (CS3) \quad \exists \vdash_{\bar{X}} \varphi = \varphi. \]

The language of \( LP_s \) is the language of \( L_s \) with extra predicate symbols \( Pr_{\tau, s}(z, \bar{X}) \), where both \( \varphi \) and \( \bar{X} \) belong to the language of \( L_s \). Axioms of the theory \( LP_s \) are all axioms of \( LP \), which are formulas of the language \( LP_s \).

If \( (A) \) is a formula of \( LP \), we denote \( (A_s) \) the version of \( (A) \) with types not greater than \( s \).

4.3. Axiomatic Theory \( SLP \)

We can introduce finite sequences in \( LP \) and for any \( k \)-functional \( F \) use the notation \( F(n) = < F(0), ..., F(n - 1) > \). We consider the following three axioms for lawless functionals.
(LL1) The axiom of existence of lawless functionals:
\[
\forall x \forall y \exists \delta^x (\forall y \leq x)(\delta^y(y) = F(y)).
\]

(LL2) \(\delta^u \cap \delta^v \neq \emptyset\).

(LL3) The principle of open data:
\[
\phi_{\mathcal{S}_P}(x) \supset \exists y \forall z \exists w \phi_{\mathcal{S}_P}(x, y) \supset \phi_{\mathcal{S}_P}(z) \supset \phi_{\mathcal{S}_P}(w).
\]

where \(\phi\) is a formula of \(\mathcal{L}_P\), \(\text{sort}(\phi) \leq n\) and \(\phi\) does not have non-lawlike parameters of type \(n\) other than \(\mathcal{X}_n\).

Denote (LL) a conjunction of the closures of (LL1), (LL2) and (LL3).

For an \(m\)-functional \(F\) of language \(\mathcal{L}_P\) and \(n \leq m\) we denote \(F(0) \ldots (0)\) as \(F(0)^n\) for brevity. We consider the following two choice axioms.

(C1) The axiom of choice for numbers:
\[
\forall x \exists y \phi(x, y) \supset \exists F^x \forall y \phi(x, F^x(y)).
\]

where \(\phi\) is a formula of \(\mathcal{L}_P\) and \(m \geq \max(\text{sort}(\phi), 1)\).

(C2) The axiom of choice with uniqueness:
\[
\forall x \exists F^x \forall y \phi(x, F^x(y)) \supset \exists F^x \forall y \phi(x, F^x(y)) \supset \forall y \phi(x, y).
\]

where \(\phi\) is a formula of \(\mathcal{L}_P\) and \(m \geq \max(\text{sort}(\phi), n+1)\).

Denote \(C\) a conjunction of the closures of (C1) and (C2).

Bar induction axiom:

\[
\forall F \exists x \phi(F(x)) \land \forall x \forall y [\phi(y) \supset \phi(y \ast (x))]<
\land \forall y [\forall x \phi(y \ast (x)) \supset \psi(x)] \land \forall y [\phi(y) \supset \psi(y)] \supset \psi(0).
\]

Axiomatic theory \(\mathcal{L}_P\) is defined by:
\[
\mathcal{L}_P = \mathcal{L}_P + (LL) + (C) + (BI).
\]

In the name of the theory \(\mathcal{L}_P\), \(S\) stands for strong, \(L\) for lawless and \(P\) for proof (relating to the "creating subject").

The fragment \(\mathcal{L}_P^s\) is defined by:
\[
\mathcal{L}_P^s = \mathcal{L}_P^s + (LL) + (C) + (BI).
\]

5. Application of the Topological Beth Model

As an application, we construct a Beth model for any fragment \(\mathcal{L}_P^s\) of the theory \(\mathcal{L}_P\). This is sufficient for the proof of consistency of \(\mathcal{L}_P\), since any formal proof in the theory \(\mathcal{L}_P\) is finite and therefore it is a proof in some fragment \(\mathcal{L}_P^s\).

5.1. Notations

A sequence of elements \(x_0, x_1, \ldots, x_n\) is denoted \(x = (x_0, x_1, \ldots, x_n)\) and we denote \(\text{lib}(x) = n + 1\).

The symbol \(*\) denotes the concatenation function:
\[
\langle x_0, x_1, \ldots, x_n \rangle * \langle y_0, y_1, \ldots, y_m \rangle = \langle x_0, x_1, \ldots, x_n, y_0, y_1, \ldots, y_m \rangle.
\]

For a sequence \(x\), \(\langle x \rangle_n\) denotes its \(n\)th element and \(\overline{x}_n\) denotes the initial segment of \(x\) of length \(n\) for any \(n < \text{lib}(x)\). So if \(x = (x_0, x_1, \ldots, x_{n-1})\) and \(n \leq m\), then \(\overline{x}_n = (x_0, x_1, \ldots, x_{n-1})\).

For a function \(f\) on natural numbers, \(\overline{f}(n)\) denotes the initial segment of \(f\) of length \(n\), that is the sequence \((f(0), f(1), \ldots, f(n-1))\).

For a function \(f\) of two variables denote
\[
f^{(0)} = \lambda y. f(x, y).
\]

Next we introduce a few notations for a fixed set \(b\).

1. \(b^{(n)}\) is the set of all sequences of elements of \(b\) that have length \(n\).
2. \(b^*\) is the set of all finite sequences of elements of \(b\).
3. On the set \(b^\ast\) a partial order \(\leq\) is defined by the following:
\[
y \leq x \text{ if } x \text{ is an initial segment of } y.
\]

With this order \(b^\ast\) is a tree growing down; its root is the empty sequence \(\varepsilon\).

Suppose \(\langle d, \leq\rangle\) is a partially ordered set and \(f: (d \times d) \to c\).

1. \(f\) is called monotonic on \(d\) if for any \(x, y \in d\), \(y \leq x \Rightarrow f^{(0)} \subseteq f^{(0)}\).
2. \(f\) is called complete on \(d\) if for any path \(S\) in \(d\), \(\bigcup \{f^{(0)} | x \in S\}\) is a total function on \(\omega\).

5.2. Beth Model \(\mathcal{B}_1\) for the Language \(L_1\)

Fix \(s \geq 1\). To construct a Beth model \(\mathcal{B}_1\) for the language \(L_1\), we will specify a tree \(\langle M, \preceq\rangle\), a domain function \(\text{Cast}\), \(\text{Fn}\) and \(\text{Pr} \).

1) First we introduce a triple of objects \(\langle a_0, d_0, \preceq\rangle\) by induction on \(k\); here \(\preceq\) is a partial order on \(d_k\).
• $a_0 = \omega$, the set of all natural numbers
• $d_0 = \{(x) | x \in a_0^*\}$; thus, each element of $d_0$ is a sequence of length 1
• $\preceq_0$ is generated by the order $\preceq$ on $a_0^*$

$\langle x \rangle \preceq_0 \langle y \rangle$ if $x \preceq y$, that is the sequence $y$ is an initial segment of the sequence $x$.

For $\langle x \rangle \in d_0$ we denote $lh(\langle x \rangle) = lh(x)$ and call it the length of $\langle x \rangle$.

For $k \geq 1$:

• $a_k = \{ f | f : (d_{k-1} \times \omega)^{< \kappa} \rightarrow a_{k-1} \}$ and $f$ is complete and monotonic;
• $d_k = \bigcup_{n=0}^{\kappa} a_k^{(n)} \times a_k^{(n)} \times \cdots \times a_k^{(n)}$;
• $\preceq_k$ is defined by:

$x \preceq_k y$ if for each $i \leq k$, $\langle x \rangle_i \preceq (\langle y \rangle)_i$, that is the sequence $\langle y \rangle$ is an initial segment of the sequence $\langle x \rangle$.

If $x \in a_k^{(m)} \times \cdots \times a_k^{(m)}$, we denote $lh(x) = m$ and call it the length of $x$.

We take $M = d_{\kappa}$ with partial order $\preceq_{\kappa}$, which we denote just $\preceq$.

With this order $M$ is a subtree of the direct product of trees $a_0^*, \ldots, a_{\kappa-1}^*$. $M$ is a non-countable tree growing down. Its root is $\varepsilon = \langle < , , , < , , , > \rangle$.

1) Next we define a domain for each sort of variables.

a) Domain for natural numbers is $a_0 = \omega$.
b) Domain for $k$-functionals is $a_k (k = 1, 2, \ldots, s)$.
c) Domain for lawlike $k$-functionals is

$b_k = \{ f \in a_k | f^{(\omega)} \text{ is a total function} \}$.
d) Before defining a domain for lawless functionals we introduce an auxiliary set $c_k (k \geq 1)$:

$c_k = \{ \xi | \xi : (a_0 \times a_0) \rightarrow a_{k-1} \text{ and } \forall n (\xi^{(n)} \text{ is a bijection on } a_k) \}$.

Elements of $c_k$ are called $k$-permutations. For any $k$-permutation $\xi$ we define $v_k (\xi)$ as the function $f \in a_k$ such that for any $n \in \omega, x \in d_{k-1}$:

\[
f(x, n) = \begin{cases} \xi^{(n)} (\xi^{(n-1)}) & \text{if } n \leq lh(x), \\ \text{undefined otherwise.} & \end{cases}
\]

Domain for lawless $k$-functionals is:

$l_k = \{ v_k (\xi) | \xi \in c_k, k = 1, 2, \ldots, s \}$.

Clearly, $b_k \subseteq a_k$ and $l_k \subseteq a_k$, $k = 1, \ldots, s$. For any $\alpha$ and $k < s$, $lh(\alpha) = lh(\langle \alpha \rangle_k)$.

2) Next we define $\overline{C_{\mu \sigma}}$, that is interpretation $\overline{C}$ for each constant $C$.

a) $\overline{0} = 0$.
b) The interpretation of constant $K^k (k = 1, \ldots, s)$ is given by the following:

$K^k (x, n) = 0$ for any $n \in \omega$ and $x \in d_{\kappa}$;

$K^{k+1} (x, n) = K^k$ for any $n \in \omega$ and $x \in d_{k}$.

3) Next we define $\overline{F}$, $n$. This means defining a partial function $\overline{H}^{[\alpha]}$ for each functional symbol $h$ and $\alpha \in M$.

a) If $\theta$ is $+ \text{ or } \cdot$, then $\overline{\theta}^{[\alpha]} = \theta$.
b) A successor function $S^\omega$ of type $k (k = 0, 1, \ldots, s)$ is defined by:

$S^\omega (x) = x + 1$; $S^{\omega+1} (f) = S^\omega \circ f$.

Functional symbol $S$ is interpreted by: $S^{[\alpha]} = S^0$.

Functional symbol $N^k$ is interpreted by: $N^{[\alpha]} = N^k$.

c) Functional symbol $Ap^\omega$ is interpreted by:

$Ap^{[\alpha]}_n (f, n) = \overline{f} (\overline{a} (k), n)$ for any $n \in \omega, f \in a_k$, $k = 1, \ldots, s$.

Thus, at each node $\alpha$ the values of a $k$-functional $f$ depend not on the entire $\alpha$ but only on its first $k$ components.

Due to the definition of $a_0$, the conditions (4) and (5) of the Definition 3.9 of Beth model are satisfied.

4) The only predicate symbols in the language $L$ are equalities $=_{k} (k = 0, 1, 2\ldots)$.

$\overline{Pr}$ is defined by:

$\overline{Pr} (p = q) = \begin{cases} T & \text{if } p = q, \\ \bot & \text{otherwise.} \end{cases}$

For any $p, q \in a_0$. Clearly, $\overline{Pr} (p = q)$ is an open set.

This completes the definition of the model $\overline{\beta_{\mu}}$ for the language of $L_{\mu}$. 221
For the general case of Beth model we already defined:
- $Z^{[\alpha]}$, where $Z$ an evaluated term or $n$-functional (Definition 3.10) and
- the forcing relation $\alpha \Vdash \phi$ (Definition 3.14)

5.3. Extending the Model $\beta_\delta$ to the Language SLP

First we extend the model to the language $LP_n$. For that we need to interpret the extra predicate symbols $P_{\bar{y}}(\bar{z},\bar{X})$, where both $\phi$ and $X$ belong to the language of $L_\delta$.

**Definition 5.1**

Consider a formula $\phi$ of $L_\delta$ with all its parameters in the list $X_1, \ldots, X_k$, which is denoted $\bar{X}$ in short. For brevity we denote the predicate $P_{\bar{y}}(\bar{z},\bar{X})$ as $P_{\bar{y}}(\bar{z},\bar{X})$. We define:

$$\mathcal{F}(P_{\bar{y}}(n,\bar{q})) = \{a | \exists \gamma[\alpha \leq \gamma \& lh(\gamma) = n \& \gamma \Vdash \phi(\bar{q})]\},$$

where, $\bar{q}$ is the list $q_1, \ldots, q_k$.

Let us show that $\mathcal{F}(P_{\bar{y}}(n,\bar{q}))$ is an open set in the order topology. Suppose $a \in \mathcal{F}(P_{\bar{y}}(n,\bar{q}))$. Then there is $\gamma$ such that $\alpha \leq \gamma \& lh(\gamma) = n \& \gamma \Vdash \phi(\bar{q})$. Then for any $\beta \leq \alpha$ we have $\beta \leq \gamma$, so $\beta \in \mathcal{F}(P_{\bar{y}}(n,\bar{q}))$.

The language of SLP is the same as the language $LP_n$, so the extended model $\beta_\delta$ is also a Beth model for the language of SLP.

**Lemma 5.2**

$$\alpha \Vdash P_{\bar{y}}(n,\bar{q}) \iff (\forall S \in Q(a))(\exists \gamma \in S)[lh(\gamma) = n \& \gamma \Vdash \phi(\bar{q})].$$

**Proof**

Proof follows from the definitions.

5.4. Soundness of the Model $\beta_\delta$ for the Theory SLP

**Theorem 5.3. Soundness Theorem.**

$$SLP_{\alpha} \vdash \phi \Rightarrow \beta_\delta \Vdash \phi.$$  

**Proof**

Proof is by induction on the length of derivation of $\phi$.

Here we provide proofs only for the case of the axioms (CS1) - (CS3) for the "creating subject". We prove these in all detail to illustrate our application of the topological Beth model. Other, more technical proofs can be found in (Kachapova, 2014; 2015). We use the notations of Definition 5.1.

(CS1) is ($\rightarrow \phi \rightarrow \neg (\neg \phi)$). It is sufficient to prove:

$$\varepsilon \Vdash P_{\bar{y}}(n,\bar{q}) \rightarrow \neg P_{\bar{y}}(n,\bar{q}).$$

By Theorem 3.15.5) it is equivalent to:

$$((\forall S \in Q(a))(\exists \alpha \in S)[\alpha \Vdash P_{\bar{y}}(n,\bar{q}) \text{ or } \alpha \Vdash \neg P_{\bar{y}}(n,\bar{q})].$$

Consider a path $S \in Q(a)$. Fix $\alpha \in S$ with $lh(\alpha) = n$.

There are two cases: $\alpha \Vdash \phi(q)$ or $\alpha \nvdash \phi(q)$ (we use classical logic in metamathematics).

Case 1. $\alpha \Vdash \phi(\bar{q})$.

Since $lh(\alpha) = n$, by Lemma 5.2 we have $\alpha \Vdash P_{\bar{y}}(n,\bar{q})$.

Case 2. $\alpha \nvdash \phi(\bar{q})$.

We will show that in this case $\alpha \nvdash \neg P_{\bar{y}}(n,\bar{q})$. By Theorem 3.15.8), it is equivalent to:

$$(\forall \beta \leq \alpha) \neg (\beta \Vdash P_{\bar{y}}(n,\bar{q})).$$

Suppose $\beta \Vdash P_{\bar{y}}(n,\bar{q})$ for some $\beta \leq \alpha$. Consider an arbitrary $S \in Q(\beta)$. By Lemma 5.2, there is $\gamma \in S$ such that $lh(\gamma) = n \& \gamma \Vdash \phi(\bar{q})$.

Since $S \in Q(\beta)$ and $\beta \leq \alpha$, we have $\alpha \in S_1$. Both $\gamma \in S_1$ and $lh(\alpha) = n = lh(\gamma)$, so $\alpha = \gamma$ and $\alpha \Vdash \phi(\bar{q})$.

Contradiction. Therefore $\neg (\beta \Vdash P_{\bar{y}}(n,\bar{q}))$.

**Proof**

$$\beta_\delta \Vdash (CS2).$$

(CS2) is ($\rightarrow \phi \rightarrow \neg (\neg \phi \rightarrow \phi$). It is sufficient to prove:

$$\varepsilon \Vdash P_{\bar{y}}(n,\bar{q}) \rightarrow P_{\bar{y}}(n \& m,\bar{q}).$$

Fix $\alpha$. Suppose $\alpha \Vdash P_{\bar{y}}(n,\bar{q})$. Consider $S \in Q(a)$. By Lemma 5.2, there is $\gamma \in S$ such that $lh(\gamma) = n$ and $\gamma \Vdash \phi(\bar{q})$.

There is $\beta \in S$ with $lh(\beta) = n \& m$. Since $\beta \leq \gamma$, we have $\beta \Vdash \phi(\bar{q})$ by monotonicity of forcing (Theorem 3.15.1)). This proves $\alpha \Vdash P_{\bar{y}}(n \& m,\bar{q})$.
Proof of $\beta_3 \vdash (CS3)$.

(CS3) is $\exists z (\vdash z \varphi) \equiv \varphi$.

It is sufficient to prove: $\varepsilon \vdash \exists z P_0(z, \overline{q}) \equiv \varphi(\overline{q})$,

which is equivalent to

$\varepsilon \vdash \exists z P_0(z, \overline{q}) \supset \varphi(\overline{q})$  \hspace{1cm} (7)

and

$\varepsilon \vdash \varphi(\overline{q}) \supset \exists z P_0(z, \overline{q})$.  \hspace{1cm} (8)

6. Conclusion

In this study we describe detailed steps of construction of Beth model from the topological point of view. The model is applied to intuitionistic functionals of high types and the theory SLP that includes lawless functionals, “the creating subject”, bar induction and some choice axioms; the model is used to prove the consistency of SLP. The topological version of Beth model and our model, in particular, can potentially be useful to investigate other intuitionistic principles with respect to functionals of high types.

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Ethics

This is a mathematical article; no ethical issues can arise after its publication.

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