

Analytical Investigation of Hyperbolic Equations via He's Methods

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Abstract: Perturbation methods depend on a small parameter which is difficult to be found for real-life nonlinear problems. To overcome this shortcoming, two new but powerful analytical methods were introduced to solve nonlinear heat transfer problems in this letter, one is He's Variational Iteration Method (VIM) and the other is the Homotopy-Perturbation Method (HPM). Nonlinear hyperbolic equations were used as examples to illustrate the simple solution procedures. These methods were useful and practical for solving the nonlinear hyperbolic equation, which is associated with variable initial condition. Comparison of the results has been obtained by both methods with exact solutions reveals that both methods were tremendously effective.

Key words: Hyperbolic equations, homotopy perturbation method, variation iteration method

INTRODUCTION

Hyperbolic partial differential equations are the subject of many researches because of their application in many engineering fields such as wave equation and telegraph equation. In recent years, several such techniques have drawn special attention, such as Hirtoa's bilinear method^[5], the Adomian's decomposition method^[21], the EXP function method^[22-26], fractional method^[27-33], the Homotopy Perturbation Method (HPM)^[3,34-38] and Variational Iteration Method (VIM)^[39-47]. Various methods for obtaining explicit solutions to hyperbolic partial differential equations have been proposed^[1-8].

Biazar *et al.*^[1] use Adomian decomposition method to solve this equation. In this work we use homotopy perturbation method and variational iteration method to solve hyperbolic equations. Unlike classical techniques, the nonlinear equations are solved easily and elegantly without transforming or linearizing the equation by using the Homotopy Perturbation Method (HPM)^[9-11]. It provides an efficient explicit solution with high accuracy, minimal calculations and avoidance of physically unrealistic assumptions.

The HPM was first proposed by He^[12-16] and has been shown to solve a large class of nonlinear problems effectively, easily and accurately with approximations converging rapidly to accurate solutions. The HPM was proposed to search for limit cycles or bifurcation curves of nonlinear equations^[17]. In^[14], a heuristic example was given to illustrate the basic idea of the HPM and its

advantages over the d-method, the method was also applied to solve boundary value problems^[18] and heat radiation equations^[19]. Variation iteration method is based on the use of Lagrange multipliers for identification of optimal values of parameters in a functional. Using this method a rapid convergent sequence is produced. The variational iteration method is suitable for finding the approximation of the solution without discretization of the problem^[20].

The general form of hyperbolic equation as a kind of second-order quasi-linear partial differential equation is:

$$a \frac{\partial^2 u}{\partial x^2} + b \frac{\partial^2 u}{\partial x \partial y} + c \frac{\partial^2 u}{\partial y^2} + e = 0 \quad (1)$$

Where a, b, c and e may be functions of x, y, u, $\frac{\partial u}{\partial x}$ and

$\frac{\partial u}{\partial y}$ but not of $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial x \partial y}$ and $\frac{\partial^2 u}{\partial y^2}$, i.e., The second order derivatives occur only to the first degree. If $b^2 - 4ac > 0$, then Eq. 1 is called hyperbolic equation.

MATERIALS AND METHODS

In this study, we have applied the Homotopy perturbation method and variational iteration method to the discussed problems. To illustrate the basic ideas of methods, we have considered the following nonlinear differential equation.

Homotopy perturbation method: To illustrate the basic ideas of this method, we consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, r \in \Omega \quad (2)$$

With the boundary conditions of:

$$B(u, \frac{\partial u}{\partial n}) = 0, r \in \Gamma \quad (3)$$

Where A, B, f (r) and G are a general differential operator, a boundary operator a known analytical function and the boundary of the domain Ω , respectively.

Generally speaking, the operator A can be divided into a linear part L and a nonlinear part N. Eq. 2 can therefore be rewritten as:

$$L(u) + N(u) - f(r) = 0 \quad (4)$$

By the homotopy technique, we construct a homotopy $v(r,p): \Omega \times [0, 1] \rightarrow \mathcal{R}$ which satisfies:

$$\begin{aligned} H(v,p) &= (1-p)[L(v) - L(u_0)] + \\ p[A(u) - f(r)] &= 0, p \in [0,1], r \in \Omega \end{aligned} \quad (5)$$

or

$$\begin{aligned} H(v,p) &= L(v) - L(u_0) + pL(u_0) + \\ p[N(v) - f(r)] &= 0 \end{aligned} \quad (6)$$

Where $p \in [0, 1]$ is an embedding parameter, while u_0 is an initial approximation of Eq. 2 which satisfies the boundary conditions.

Obviously, considering Eq. 5 and 6 we will have:

$$H(v, 0) = L(v) - L(u_0) = 0 \quad (7)$$

$$H(v, 1) = A(v) - f(r) = 0 \quad (8)$$

The changing process of p from zero to unity is just that of $v(r, p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation and $L(v) - L(u_0)$ and $A(v) - f(r)$ are being called homotopy.

According to the HPM, we can first use the embedding parameter p as a small parameter and assume that the solution of Eq. 5 and 6 can be written as a power series in p, namely:

$$v = v_0 + pv_1 + p^2v_2 + \dots \quad (9)$$

Setting $p = 1$ result in the approximate solution of Eq. 2 to:

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots \quad (10)$$

The combination of the perturbation method and the homotopy method is called the HPM, which eliminates the drawbacks of the traditional perturbation methods. The series 10 is convergent for most cases. However, the convergent rate depends on the nonlinear operator A (v) (the following opinions are suggested by He^[3]):

- The second derivative of N (v) with respect to V must be small because the parameter may be relatively large, i.e., $p \rightarrow 1$
- The norm of $L^{-1} \frac{\partial N}{\partial v}$ must be smaller than one so that the series converges

Variational iteration method: To clarify the basic ideas of VIM, we consider the following differential equation:

$$Lu + Fu = g(t) \quad (11)$$

Where L is a linear operator, F is a nonlinear operator and g (t) is a heterogeneous term.

According to VIM, we can write down a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda (Lu_n(\tau) + F\tilde{u}_n(\tau) - g(\tau)) d\tau \quad (12)$$

Where λ is a general Lagrangian multiplier^[17-20] which can be identified optimally via the variational theory.

The subscript n indicates the nth approximation and \tilde{u}_n is considered as a restricted variation^[17-20], i.e., $\delta \tilde{u}_n = 0$.

Applications: In order to assess the accuracy of VIM and HPM for solving hyperbolic equations and to compare it with exact solution, we will consider the three following examples.

Example1: Consider the partial differential equation with the initial conditions^[7]:

$$\frac{\partial^2 u}{\partial x^2} + (1 - 2x) \frac{\partial^2 u}{\partial x \partial y} + (x^2 - x - 2) \frac{\partial^2 u}{\partial y^2} = 0 \quad (13)$$

$$u(x, 0) = x, \\ \frac{\partial u(x, 0)}{\partial y} = 1$$

Homotopy perturbation method: Now we apply homotopy perturbation to Eq. 13:

$$H(u, p) := (1-p)(x^2 - x - 2) \left[\frac{\partial^2}{\partial y^2} v(x, y) \right] + p \left[\frac{\partial^2}{\partial x^2} v(x, y) + (1-2x) \left[\frac{\partial^2}{\partial y \partial x} v(x, y) \right] + (x^2 - x - 2) \left[\frac{\partial^2}{\partial y^2} v(x, y) \right] \right] = 0 \quad (14)$$

Substituting Eq. 9 into Eq. 14 and rearranging based on powers of p-terms, we have:

$$p^0: x^2 \left[\frac{\partial^2}{\partial y^2} v_0(x, y) \right] - x \left[\frac{\partial^2}{\partial y^2} v_0(x, y) \right] - 2 \left[\frac{\partial^2}{\partial y^2} v_0(x, y) \right] = 0 \quad (15)$$

$$p^1: + \frac{\partial^2}{\partial y \partial x} v_0(x, y) - 2x \left[\frac{\partial^2}{\partial y \partial x} v_0(x, y) \right] - x \left[\frac{\partial^2}{\partial y^2} v_1(x, y) \right] - 2x \left[\frac{\partial^2}{\partial y \partial x} v_1(x, y) \right] + x^2 \left[\frac{\partial^2}{\partial y^2} v_1(x, y) \right] = 0 \quad (16)$$

$$p^2: + \frac{\partial^2}{\partial x^2} v_1(x, y) - 2 \left[\frac{\partial^2}{\partial y^2} v_2(x, y) \right] + \frac{\partial^2}{\partial y \partial x} v_1(x, y) - x \left[\frac{\partial^2}{\partial y^2} v_2(x, y) \right] = 0 \quad (17)$$

$$p^3: + \frac{\partial^2}{\partial y \partial x} v_2(x, y) - 2 \left[\frac{\partial^2}{\partial y^2} v_3(x, y) \right] + x^2 \left[\frac{\partial^2}{\partial y^2} v_3(x, y) \right] - x \left[\frac{\partial^2}{\partial y^2} v_3(x, y) \right] = 0 \quad (18)$$

To determine u, the above equations should be solved. Considering the appropriate initial conditions we have:

$$v_0(x, 0) = x + y, \frac{\partial}{\partial y} v_0(x, 0) = 1 \quad (19)$$

$$v_i(x, 0) = 0, \frac{\partial}{\partial y} v_i(x, 0)|_{y=0} = 0, i = 1, 2, \dots$$

The Solution of Eq. 15-19 may be written as follows:

$$v_0(x, 0) = x + y \quad (20)$$

In the same manner, the rest of components were obtained using the maple package.

According to the HPM, we can conclude:

$$u(x, y) = \lim_{p \rightarrow 1} v(x, y) = v_0(x, y) + v_1(x, y) + v_2(x, y) + \dots \quad (21)$$

In this manner three components of the perturbation series 9 were obtained. So, we have:

$$u(x, y) = x + y$$

Variational iteration method: Now Before applying this procedure to Eq. 13 following^[42,43], we construct a correction functional, as follows:

$$u_{n+1}(x, y) = u_n(x, y) + \int_0^y \lambda(\tau) \left[\frac{\partial^2}{\partial x^2} u_n(x, \tau) + (1-2x) \left[\frac{\partial^2}{\partial x \partial \tau} u_n(x, \tau) \right] + (x^2 - x - 2) \left[\frac{\partial^2}{\partial \tau^2} u_n(x, \tau) \right] \right] d\tau \quad (22)$$

Its stationary conditions can be obtained as follows:

$$1 - (x^2 - x - 2) \left[\frac{d}{dy} \lambda(y) \right] = 0, \\ (x^2 - x - 2) \left[\frac{d^2}{d\tau^2} \lambda(\tau) \right] = 0, \\ \lambda(y) \left[\frac{\partial^2}{\partial x^2} u_n(x, y) \right] = 0, \lambda(y) = 0 \quad (23)$$

The Lagrangian multiplier can therefore be identified as:

$$\lambda(\tau) = \frac{\tau - y}{x^2 - x - 2} \quad (24)$$

As a result, we obtain the following iteration formula:

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^y \lambda(\tau) \left[\frac{\partial^2}{\partial x^2} u_n(x,\tau) + (1-2x) \left[\frac{\partial^2}{\partial x \partial \tau} u_n(x,\tau) \right] + (x^2 - x - 2) \left[\frac{\partial^2}{\partial \tau^2} u_n(x,\tau) \right] \right] d\tau \quad (25)$$

Now we start with an arbitrary initial approximation that satisfies the initial condition:

$$u_0(x,y) = x + y \quad (26)$$

Using the above variational formula 25, we have:

$$u_1(x,y) = u_0(x,y) + \int_0^y \lambda(\tau) \left[\frac{\partial^2}{\partial x^2} u_0(x,\tau) + (1-2x) \left[\frac{\partial^2}{\partial x \partial \tau} u_0(x,\tau) \right] + (x^2 - x - 2) \left[\frac{\partial^2}{\partial \tau^2} u_0(x,\tau) \right] \right] d\tau \quad (27)$$

Substituting Eq. 26 in to Eq. 27 and after simplifications, we have:

$$u(x,y) = x + y \quad (28)$$

In the same way, we can obtain $U_2(X, Y)$, $U_3(X, Y)$ and the rest of the components of the iteration formula.

Example 2 Consider the partial differential equation with the initial conditions^[7]:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} - 2 \frac{\partial^2 u}{\partial y^2} + 1 &= 0, \\ u(x,0) &= x, \\ \frac{\partial u(x,0)}{\partial y} &= x. \end{aligned} \quad (29)$$

Homotopy perturbation method: By applying homotopy perturbation to Eq. 29. We have:

$$\begin{aligned} H(u,p) &= -2(1-p) \left[\frac{\partial^2}{\partial y^2} v(x,y) \right] + p \left[\frac{\partial^2}{\partial x^2} v(x,y) \right. \\ &\left. + \frac{\partial^2}{\partial y \partial x} v(x,y) - 2 \left[\frac{\partial^2}{\partial y^2} v(x,y) \right] + 1 \right] = 0 \end{aligned} \quad (30)$$

Substituting Eq. 9 into Eq. 29 and rearranging based on powers of p-terms, we have:

$$p^0 : -2 \left[\frac{\partial^2}{\partial y^2} v_0(x,y) \right] = 0 \quad (31)$$

$$p^1 : -2 \left[\frac{\partial^2}{\partial y^2} v_1(x,y) \right] + \frac{\partial^2}{\partial x^2} v_0(x,y) + \frac{\partial^2}{\partial y \partial x} v_0(x,y) + 1 = 0 \quad (32)$$

$$v_0(x,y) + \frac{\partial^2}{\partial y \partial x} v_0(x,y) + 1 = 0$$

$$p^2 := -2 \left[\frac{\partial^2}{\partial y^2} v_2(x,y) \right] + \frac{\partial^2}{\partial x^2} v_1(x,y) + \frac{\partial^2}{\partial y \partial x} v_1(x,y) = 0 \quad (33)$$

$$p^3 := -2 \left[\frac{\partial^2}{\partial y^2} v_3(x,y) \right] + \frac{\partial^2}{\partial y^2} v_2(x,y) + \frac{\partial^2}{\partial y \partial x} v_2(x,y) = 0 \quad (34)$$

To determine u, the above equations should be solved. Considering the appropriate initial conditions we have:

$$\begin{aligned} v_0(x,0) &= x + xy, \quad \frac{\partial}{\partial y} v_0(x,0) = x, \\ v_i(x,0) &= 0, \quad \frac{\partial}{\partial y} v_i(x,0) \Big|_{y=0} = 0, i=1,2,\dots \end{aligned} \quad (35)$$

The solution of Eq. 31-35 may be written as follows:

$$n_0(x,y) = x + xy \quad (36)$$

$$n_1(x,y) = \frac{1}{2} y^2 \dots \text{andsoon} \quad (37)$$

In the same manner, the rest of components was obtained using the maple package. According to the HPM, we can conclude:

$$u(x,y) = \lim_{p \rightarrow 1} v(x,y) = v_0(x,y) + v_1(x,y) + v_2(x,y) + \dots, \quad (38)$$

Therefore the solution is:

$$u(x,y) = x + xy + \frac{1}{2} y^2 \quad (39)$$

Variational iteration method: First we construct a correction functional like example 1, as follows:

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^y \lambda(\tau) \left[\frac{\partial^2}{\partial x^2} u_n(x,\tau) + \frac{\partial^2}{\partial x \partial \tau} u_n(x,\tau) - 2 \left(\frac{\partial^2}{\partial \tau^2} u_n(x,\tau) \right) + 1 \right] d\tau \quad (40)$$

Its stationary conditions can be obtained as follows:

$$1 + 2\left(\frac{d}{dy}\lambda(y)\right) = 0, -2\left(\frac{d^2}{d\tau^2}\lambda(\tau)\right) = 0, \tag{41}$$

$$\lambda(y)\left[\frac{\partial^2}{\partial x^2}u_n(x,y)\right] = 0, \lambda(y) = 0$$

The Lagrangian multiplier can therefore be identified as:

$$\lambda(\tau) = \frac{\tau - y}{-2} \tag{42}$$

As a result, we obtain the following iteration formula:

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^y \lambda(\tau)\left[\frac{\partial^2}{\partial x^2}u_n(x,\tau) + \frac{\partial^2}{\partial x \partial \tau}u_n(x,\tau) - 2\left(\frac{\partial^2}{\partial \tau^2}u_n(x,\tau)\right) + 1\right]d\tau \tag{43}$$

Now we start with an arbitrary initial approximation that satisfies the initial condition:

$$u_0(x,y) = xy + x \tag{44}$$

Using the above variational formula 43, we have

$$u_1(x,y) = u_0(x,y) + \int_0^y \lambda(\tau)\left[\frac{\partial^2}{\partial x^2}u_0(x,\tau) + \frac{\partial^2}{\partial x \partial \tau}u_0(x,\tau) - 2\left(\frac{\partial^2}{\partial \tau^2}u_0(x,\tau)\right) + 1\right]d\tau \tag{45}$$

Substituting Eq. 44 in to Eq. 45 and after simplifications, we have:

$$u_1(x,y) = xy + x + \frac{1}{4}y^2 \tag{46}$$

In the same way, we can obtain $U_2(X, Y)$ as follows:

$$u_2(x,y) = xy + x + \frac{1}{2}y^2 \tag{47}$$

And so on. In the same manner the rest of the components of the iteration formula can be obtained.

Example 3: Consider the partial differential equation with the initial conditions^[6]:

$$\frac{\partial^2 u}{\partial x^2} - 4x^2 \frac{\partial^2 u}{\partial y^2} = 0, \tag{48}$$

$$u(x,0) = x^2,$$

$$\frac{\partial u(x,0)}{\partial y} = 0$$

Homotopy perturbation method: By applying homotopy perturbation to Eq. 48, we have:

$$H(u,p) = -4(1-p)x^2\left[\frac{\partial^2}{\partial y^2}v(x,y)\right] + p\left[\frac{\partial^2}{\partial x^2}v(x,y) - 4x^2\left[\frac{\partial^2}{\partial y^2}v(x,y)\right]\right] = 0 \tag{49}$$

Substituting Eq. 9 into Eq. 48 and rearranging based on powers of p-terms, we have:

$$p^0 : -4x^2\left[\frac{\partial^2}{\partial y^2}v_0(x,y)\right] = 0 \tag{50}$$

$$p^1 : -4x^2\left[\frac{\partial^2}{\partial y^2}v_1(x,y)\right] + \frac{\partial^2}{\partial x^2}v_0(x,y) = 0 \tag{51}$$

$$p^2 : -4x^2\left[\frac{\partial^2}{\partial y^2}v_2(x,y)\right] + \frac{\partial^2}{\partial x^2}v_1(x,y) = 0 \tag{52}$$

$$p^3 : -4x^2\left[\frac{\partial^2}{\partial y^2}v_3(x,y)\right] + \frac{\partial^2}{\partial x^2}v_2(x,y) = 0 \tag{53}$$

$$p^4 : -4x^2\left[\frac{\partial^2}{\partial y^2}v_4(x,y)\right] + \frac{\partial^2}{\partial x^2}v_3(x,y) = 0 \tag{54}$$

To determine u, the above equations should be solved. Considering the appropriate initial conditions we have:

$$v_0(x,0) = x^2, \frac{\partial}{\partial y}v_0(x,0) = 0, \tag{55}$$

$$v_i(x,0) = 0, \frac{\partial}{\partial y}v_i(x,0)|_{y=0} = 0, i = 1,2,\dots$$

The Solution of Eq. 50-55 may be written as follows:

$$v_0(x,y) = x^2 \tag{56}$$

$$v_1(x,y) = \frac{1}{4} \frac{y^2}{x^2} \tag{57}$$

$$v_2(x,y) = \frac{1}{32} \frac{y^4}{x^6} \tag{58}$$

$$v_3(x,y) = \frac{7}{640} \frac{y^6}{x^{10}} \tag{59}$$

$$v_4(x,y) = \frac{11}{2048} \frac{y^8}{x^{14}} \tag{60}$$

Therefore the solution is:

$$u(x,y) = x^2 + \frac{1}{4} \frac{y^2}{x^2} + \frac{1}{32} \frac{y^4}{x^6} + \frac{7}{640} \frac{y^6}{x^{10}} + \frac{11}{2048} \frac{y^8}{x^{14}} \quad (61)$$

Variational iteration method: First we construct a correction functional like example 1, as follows:

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^y \lambda(\tau) \left[\frac{\partial^2}{\partial x^2} u_n(x,\tau) - 4x^2 \left(\frac{\partial^2}{\partial \tau^2} u_n(x,\tau) \right) \right] d\tau \quad (62)$$

Its stationary conditions can be obtained as follows:

$$1 + 4x^2 \left(\frac{d}{dy} \lambda(y) \right) = 0, -4x^2 \left(\frac{d^2}{d\tau^2} \lambda(\tau) \right) = 0 \quad (63)$$

$$\lambda(y) \left[\frac{\partial^2}{\partial x^2} u_n(x,y) \right] = 0, \lambda(y) = 0$$

The Lagrangian multiplier can therefore be identified as:

$$\lambda(\tau) = -\frac{1}{4} \frac{\tau}{x^2} + \frac{1}{4} \frac{y}{x^2} \quad (64)$$

As a result, we obtain the following iteration formula:

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^y \lambda(\tau) \left[\frac{\partial^2}{\partial x^2} u_n(x,\tau) - 4x^2 \left(\frac{\partial^2}{\partial \tau^2} u_n(x,\tau) \right) \right] d\tau \quad (65)$$

Now we start with an arbitrary initial approximation that satisfies the initial condition:

$$u_0(x,y) = x^2 \quad (66)$$

Using the above variational formula 65, we have:

$$u_1(x,y) = u_0(x,y) + \int_0^y \lambda(\tau) \left[\frac{\partial^2}{\partial x^2} u_0(x,\tau) - 4x^2 \left(\frac{\partial^2}{\partial \tau^2} u_0(x,\tau) \right) \right] d\tau \quad (67)$$

Substituting Eq. 66 in to Eq. 67 and after simplifications, we have:

$$u(x,y) = \frac{14x^2 + y^2}{4x^2} \quad (68)$$

In the same way, we can obtain $U_2(X, Y)$, $U_3(X, Y)$ and $U_4(X, Y)$ as follows:

$$u_2(x,y) = \frac{1}{16} \frac{16x^8 + 8x^4y^2 + 3y^4}{x^6} \quad (69)$$

$$u_3(x,y) = \frac{1}{64} \frac{64x^{12} + 48x^8y^2 + 36x^4y^4 + 63y^4}{x^{10}} \quad (70)$$

$$u_3(x,y) = x^2 + \frac{1}{4} \frac{y^2}{x^2} + \frac{1}{32} \frac{y^4}{x^6} + \frac{7}{640} \frac{y^6}{x^{10}} + \frac{11}{2048} \frac{y^8}{x^{14}} \quad (71)$$

And so on. In the same manner the rest of the components of the iteration formula can be obtained.

RESULTS

The solution of hyperbolic equations has been investigated analytically by Homotopy Perturbation Method (HPM) and Variational Iteration Method (VIM). $U(X, Y)$ for both Eq. 47 and 71 have been shown in Fig. 1 and 2 respectively. Also in Tables 1-3, for some values of x and y , results for both methods have been compared with exact solution. The results have shown that they are considerably capable of solving a wide range of hyperbolic equations.

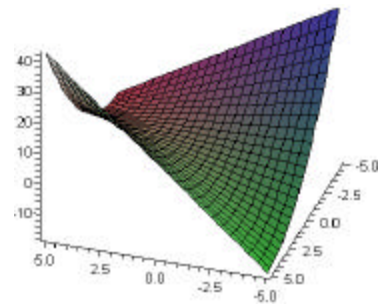


Fig. 1: 3D plot of $u(x, y)$ for Eq. 47

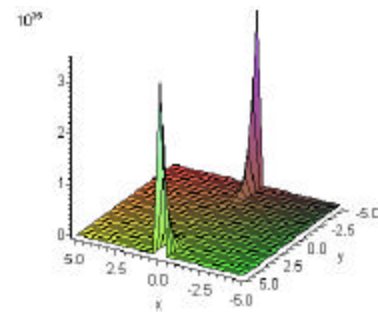


Fig. 2: 3D plot of $u(x, y)$ for Eq. 71

Table 1: The solution of $u(x, y)$ for different values of x and y

x	y	U(x, y) (HPM)	U(x, y) (VIM)	U(x, y) (exact result)
0.035	0.071	0.106	0.106	0.107
0.139	0.074	0.213	0.213	0.212
0.448	0.077	0.525	0.525	0.526
0.758	0.075	0.833	0.833	0.834
0.819	0.152	0.971	0.971	0.971

Table 2: The solution of $u(x, y)$ for different values of x and y

x	y	U(x, y) (HPM)	U(x, y) (VIM)	U(x, y) (exact result)
0.033	0.067	0.0374	0.0374	0.0378
0.133	0.067	0.1442	0.1442	0.1444
0.833	0.067	0.8911	0.8911	0.8911
0.067	0.133	0.0848	0.0848	0.0844
0.767	0.133	0.8779	0.8779	0.8778

Table 3: The solution of $u(x, y)$ for different values of x and y

x	y	U(x, y) (HPM)	U(x, y) (VIM)	U(x, y) (exact result)
0.35714	0.034286	0.12987	0.12987	0.12918
0.35357	0.035180	0.12749	0.12749	0.12709

DISCUSSION

In this study, the authors have intended to show that the two methods, HPM and VIM, are considerably capable of solving a wide range of hyperbolic equations. The examples given in this study reveal that both methods are very effective and have high accuracy. In some cases as Examples 1 and 2, after some steps the remaining terms would vanish and we derive the exact solution. In the cases as Example 3, the approximation can be obtained to any desired number of terms.

VIM and HPM do not need small parameters, the limitations and non-physical assumptions required in classical perturbation methods are eliminated, furthermore, VIM and HPM can overcome the difficulties arising in the calculation of Adomian polynomials. They do not require linearization; both methods are very promising tools for hyperbolic equations. Therefore, both methods will find applications in various fields.

CONCLUSION

The methods of HPM and VIM have been successfully performed for hyperbolic equations. In these cases, we obtained excellent performances that might lead to promising approaches for many applications. All the examples showed that the results of the present methods were in excellent agreement with exact solutions. It is capable to converge to correct results with fewest number of iterations or even once, for some cases.

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