Journal of Mathematics and Statistics 5 (4): 276-282, 2009
ISSN 1549-3644

Homomorphisms on Lattices of Measures

[1]Norris Sookoo and [2]Peter Chami
[1]University of Trinidad and Tobago, O'meara, Arima,
Trinidad, West Indies
[2]Department of Computer Science, Mathematics and Physics,
University of the West Indies, Cave Hill, Barbados, West Indies

Abstract: Problem statement: Homomorphisms on lattices of measures defined on the quotient spaces of the integers were considered. These measures were defined in terms of Sharma-Kaushik partitions. The homomorphisms were studied in terms of their relationship with the underlying Sharma-Kaushik partitions. **Approach:** We defined certain mappings between lattices of Sharma-Kaushik partitions and showed that they are homomorphisms. These homomorphisms were mirrored in homorphisms between related lattices of measures. **Results:** We obtained the structure of certain homomorphisms of measures. **Conclusion:** Further information about homomorphisms between lattices of measures of the type considered here can be obtained by investigating the underlying lattices of Sharma-Kaushik partitions.

Key words: Measure, lattice, ideal, partition

INTRODUCTION

Systems of measures having different structures and properties have long been the subject of investigation. Maharam[5] studied a family of measures with orthogonality properties and Schmidt[8] proved that a particular ordered Banach space of vector measures is a Banach lattice. Systems of measures satisfying compatibility conditions were studied by Niederreiter and Sookoo[6,7], who obtained conditions under which a partial density can be extended to a density. Sookoo and Chami[9] investigated the lattice structure of certain sets of lattices of measures defined on the quotient spaces of the integers.

We consider mappings that preserve certain elements of the structure of lattices of such measures, namely homomorphisms. We investigate some of the forms that homomorphosms may take.

The measures that we consider are defined in terms of SK-partitions of the ring of integer's module q. The studies of these partitions have been conducted by Kaushik[2-4].

We consider homomorphosms given in terms of a function defined on the class sizes of the underlying partitions. Later, we consider homomorphisms that change the number of classes or alter class sizes in a pre-determined manner.

Definitions and notations:
Notation: Let $F_q = \{0, 1, \dots, q-1\}$ be the ring of integers modulo q, $q \in \{2,3,\dots\}$.

Definition: Given F_q, $q \geq 2$, a partition $P = \{B_0, B_1, \dots, B_{m-1}\}$ of F_q is called an SK-partition if:

- $B_0 = \{0\}$ and $q-a \in B_i$ if $a \in B_i$, i = 1,2, …., m_{-1}
- If $a \in B_i$ and $b \in B_j$; $i, j = 0,1,\dots,m-1$ and if j precedes i in the order of the partition P, (written as i>j), then $\min\{a, q-a\} > \min\{b, q-b\}$.
- If i>j, ($i, j \in \{0,1,\dots,m-1\}$) and $i \neq m-1$, then:

$$|B_i| \geq |B_j| \text{ and } |B_{m-1}| \geq \frac{1}{2}|B_{m-2}|$$

where, $|B_i|$ stands for the size of the set B_i

Notation: A partition $B = \{B_0, B_1, \dots, B_{m-1}\}$ is denoted by $B = ((1, b_1, b_2, \dots, b_{m-1}))$ where $b_i = |B_i|, i = 1, 2, \dots, m-1$.

Notation: $\Im_P$ is the set of all SK-partitions.

The concept of an ideal is well known in lattice theory, Birkhoff[1].

Corresponding Author: Peter Chami, Department of Computer Science, Mathematics and Physics, University of the West Indies, Cave Hill, Barbados, West Indies

Definition: Let $(L,\le)$ be a lattice. A subset A of L is called an ideal, if:

- $a,b\in A \Rightarrow a\vee b\in A$
- $a\in A$ and $c\in L \ni c\le a \Rightarrow c\in A$

Notation: Z/qZ is the quotient group of integers modulo q with the discrete topology.

Definition: Given a partition P of F_q, we define a measure μ_P on Z/qZ as follows:

$$\mu_P(i+qZ)=j,\ \text{if } i\in B_j, i=0,1,\dots,q-1$$

Note: In this study, the SK-partitions that we consider must satisfy the condition that for each partition the class sizes never decrease as the subscript of the classes increases.

Definition: We define the class-size ordering $\le_s$ on $\Im_P$ as follows. For elements P and Q of $\Im_P \ni$:

$$P=\{B_0,B_1,\dots,B_{m-1}\}$$

And:

$$Q=\{C_0,C_1,\dots,C_{m'-1}\};m,m'\in\{2,3,4,\dots\}$$

Where:
P = An SK-partition of F_q
Q = An SK-partition of $F_{q'};q,q'\in\{2,3,\dots\}$; $q,q'\in\{2,3,\dots\}$

$P\le_s Q \Leftrightarrow \{m\le m'$ and the number of elements of F_q of weight ω with respect to P≤ the number of elements of $F_{q'}$ of weights ω with respect to Q, $\omega=0,1,\dots,m-1\}$.

Definition: Let μ_P be a measure on Z/qZ and μ_Q be a measure on $Z/q'Z$, where:

$$P=\{B_0,B_1,\dots,B_{m-1}\}$$

is an SK-partition of F_q and:

$$Q=\{C_0,C_1,\dots,C_{m'}-1\}$$

is an SK-partition of $F_{q'}$. Also, let $M_P=\{\mu_P \mid P\in\Im\}$. We define an ordering on M_P as follows: For $\mu_P,\mu_Q\in M_P$.

$\mu_P\le_\mu \mu_Q \Leftrightarrow$ {number of elements of Z/qZ of measure j≤number of elements of $Z/q'Z$ of measure j $j=0,1,\dots,m-1\}$.

Note: Clearly:

$$\mu_P\le\mu_Q \Leftrightarrow P\le_S Q$$

Remark: Clearly, from the above definition $P\le_S Q \Leftrightarrow |B_i|\le|C_i|$, i = 0,1,…,m-1.

Note: $\le_s$ is a partial ordering on $\Im_P$.

Note: Le t $m\le m'$ and:

$$A=((1,a_1,a_2,\dots,a_{m-1}))\ ,\ B=((1,b_1,b_2,\dots,b_{m'-1}))\ .$$

It is easy to show that:

$$A\vee B = ((1,\max\{a_1,b_1\},\max\{a_2,b_2\},\dots,\max\{a_{m-1},b_{m-1}\}, \max\{a_{m-1},b_m\},\max\{a_{m-1},b_{m+1}\},\dots,\max\{a_{m-1},b_{m'-1}\}))$$

$$A\wedge B = ((1,\min\{a_1,b_1\},\min\{a_2,b_2\},\dots,\min\{a_{m-1},b_{m-1}\}))\ ,$$

and that ($\Im_P,\le_s$) is a lattice.

MATERIALS AND METHODS

Function on the class sizes:
Theorem 1: Let $\phi_f:\Im_P\to\Im_P$ be defined by:

$$\phi_f[((1,g_1,g_2,\dots,g_{m-1}))]=((1,f(g_1),f(g_2),\dots,f(g_{m-1})))$$

for any element $((1,g_1,g_2,\dots,g_{m-1}))$ of $\Im_p$, where $m\in\{2,3\dots\}$ and f is a function from {2, 4, 6,…} to $\{2,4,6,\dots\}$.

ϕ_f is a lattice homomorphism if and only if f is non-decreasing.

Proof: Let ϕ_f be a lattice homomorphism on $\Im_P$ and let:

$$A,B\in\Im_P \ni$$
$$A=((1,a_1,a_2,\dots,a_{m-1}))$$
$$B=((1,b_1,b_2,\dots,b_{m'-1}))$$

where, $a_1,a_2,\dots,a_{m-1},b_1,b_2,\dots,b_{m'-1}$ are fixed, positive, even integers. We assume, without loss of generality, that $m\le m'$.

Now:

$$\phi_f(A \vee B) = \phi_f(A) \vee \phi_f(B)$$

$$\therefore \phi_f\left[\left(\left(1, a_1, a_2, ..., a_{m-1}\right)\right) \vee \left(\left(1, b_1, b_2, ..., b_{m'-1}\right)\right)\right]$$

$$= \left\{\phi\left[\left(\left(1, a_1, a_2, ..., a_{m-1}\right)\right)\right]\right\} \vee \left\{\phi_f\left[\left(\left(1, b_1, b_2, ..., b_{m'-1}\right)\right)\right]\right\}$$

$$\therefore \phi_f[((1, \max\{a_1, b_1\}, \max\{a_2, b_2\}, ..., \max\{a_{m-1}, b_{m-1}\},$$
$$\max\{a_{m-1}, b_m\}, \max\{a_{m-1}, b_{m+1}\}, ..., \max\{a_{m-1}, b_{m'-1}\}))]$$

$$= \left(\left(1, f(a_1), f(a_2), ..., f(a_{m-1})\right)\right) \vee \left(\left(1, f(b_1), f(b_2), ..., f(b_{m'-1})\right)\right)$$

$$\therefore f\left[\max\{a_1, b_1\}\right] = \max\{f(a_1), f(b_1)\}$$

$$\therefore a_1 \geq b_1 \Rightarrow f(a_1) = \max\{f(a_1), f(b_1)\}$$

$$\Rightarrow f(a_1) \geq f(b_1)$$

Since a_1 and b_1 are arbitrary, f is non-decreasing. Let f be non-decreasing. ϕ_f is clearly a function. Also:

$$\phi_f(A \vee B) = \phi_f\left[\left(\left(1, a_1, a_2, ..., a_{m-1}\right)\right) \vee \left(\left(1, b_1, b_2, ..., b_{m'-1}\right)\right)\right]$$
$$= \phi_f[1, \max\{a_1, b_1\}, \max\{a_2, b_2\}, ..., \max\{a_{m-1}, b_{m-1}\},$$
$$= \max\{a_{m-1}, b_m\}, \max\{a_{m-1}, b_{m+1}\}, ..., \max\{a_{m-1}, b_{m'-1}\}))]$$
$$= ((1, f(\max\{a_1, b_1\}), f\left(\max\{a_2, b_2\}\right), ...,$$
$$f\left(\max\{a_{m-1}, b_{m-1}\}\right), f\left(\max\{a_{m-1}, b_m\}\right),$$
$$f\left(\max\{a_{m-1}, b_{m+1}\}\right), ..., f\left(\max\{a_{m-1}, b_{m'-1}\}\right)))$$
$$= ((1, \max\{f(a_1), f(b_1)\}, \max\{f(a_2), f(b_2)\}, ...,$$
$$\max\{f(a_{m-1}), f(b_{m-1})\}, \max\{f(a_{m-1}), f(b_m)\},$$
$$= \max\{f(a_{m-1}), f(b_{m+1})\}, ..., \max\{f(a_{m-1}), f(b_{m'-1})\}))$$
$$= ((1, f(a_1), f(a_2), ..., f(a_{m-1}))) \vee$$
$$\left(\left(1, f(b_1), f(b_2), ..., f(b_{m'-1})\right)\right)$$
$$= \left[\phi_f(A)\right] \vee \left[\phi_f(B)\right]$$

In almost the same ay, we can show that:

$$\phi_f(A \wedge B) = \left[\phi_f(A)\right] \wedge \left[\phi_f(B)\right]$$

Hence ϕ_f is a lattice homorphism from $\Im_P$ to $\Im_P$.

Corollary 2: Define $\psi_f : \mu_P \to \mu_P$ by:

$$\psi_f[((1, g_1, g_2 ..., g_{m-1}))_\mu] = ((1, f(g_1), f(g_2), ..., f(g_{m-1})))_\mu$$

for any element $((1,g_1,g_2,\ldots g_{m-1}))_\mu$ of M_P. ψ_f is a lattice homomorphism iff f is non-decreasing.

Inserting classes:

Theorem 3: Let:

$$f_r : \Im_P \to \Im_P \ni f_r\left(\left(1, a_1, a_2, ..., a_{m-1}\right)\right)$$
$$= \left(\left(1, 2, 2, ..., 2, a_1, a_2, ..., a_{m-1}\right)\right)$$
$$\leftarrow \text{r twos} \rightarrow$$

for any element $\left(\left(1, a_1, a_2, ..., a_{m-1}\right)\right)$ of $\Im_P$, where r is a fixed, arbitrary element of $\{1, 2, ...\} \ni r \leq m-1$.

f_r is a homomorphism on $\Im_P$.

Proof: f_r is clearly a function.

Let:

$$A, B \in \Im_P \ni$$

$$A = \left(\left(1, a_1, a_2, ..., a_{m-1}\right)\right)$$
$$B = \left(\left(1, b_1, b_2, ..., b_{m'-1}\right)\right)$$

For fixed, arbitrary numbers $m, m' \in \{2, 3, ...\}$ $\ni m \leq m'$.

We show that:

$$f_r(A \vee B) = \left[f_r(A)\right] \vee \left[f_r(B)\right]$$
$$f_r(A \vee B) = f_r\left[\left(\left(1, a_1, a_2, ..., a_{m-1}\right)\right) \vee \left(\left(1, b_1, b_2, ..., b_{m'-1}\right)\right)\right]$$
$$= f_r[((1, \max\{a_1, b_1\}, \max\{a_2, b_2\}, ..., \max\{a_{m-1}, b_{m-1}\},$$
$$\max\{a_{m-1}, b_m\}, \max\{a_{m-1}, b_{m+1}\}, ..., \max\{a_{m-1}, b_{m'-1}\}))]$$
$$= (1, 2, 2, ..., 2, \max\{a_1, b_2\}, \max\{a_2, b_2\}, ..,$$
$$\leftarrow \text{r twos} \rightarrow$$
$$\max\{a_{m-1}, b_{m-1}\}, \max\{a_{m-1}, b_m\},$$
$$\max\{a_{m-1}, b_{m+1}\}, ..., \max\{a_{m-1}, b_{m'-1}\}))$$
$$= ((1, 2, 2, ..., 2, a_1, a_2, ..., a_{m-1})) \vee$$
$$\leftarrow \text{r twos} \rightarrow$$
$$((1, 2, 2, ..., 2, b_1, b_2, ..., b_{m'-1}))$$
$$\leftarrow \text{r twos} \rightarrow$$

$$= \left[f_r(A)\right] \vee \left[f_r(B)\right] \quad (1)$$

We now prove that:

$$f_r(A \wedge B) = \left[f_r(A)\right] \wedge \left[f_r(B)\right]$$
$$f_r(A \wedge B) = f_r[((1, a_1, a_2, ..., a_{m-1})) \wedge \left(1, b_1, b_2, ..., b_{m'-1}\right))]$$

$$= f_r[((1, \min\{a_1, b_1\}, \min\{a_2, b_2\}, ..., \min\{a_{m-1}, b_{m-1}\}))]$$
$$= ((\underbrace{1, 2, 2, ..., 2}_{\leftarrow r \text{ twos} \rightarrow}, \min\{a_1, b_1\}, \min\{a_2, b_2\}, ..., \min\{a_{m-1}, b_{m-1}\}))$$
$$= ((1, 2, 2, ..., 2, a_1, a_2, ..., a_{m-1})) \wedge ((1, 2, 2, ..., 2, b_1, b_2, ..., b_{m'-1}))$$

$$[f_r(A)] \wedge [f_r(B)] \qquad (2)$$

From (1) and (2), we conclude that f_r is a lattice homomorphism.

Remark: f_r maps an element of $\Im_{P,m}$ to an element of $\Im_{P,m+r}$ for each $m \in \{2,3,...\}$.

Corollary 4: Define $\varphi_{f_r} : M_P \to M_P$ by:

$$\varphi_{f_r}\left[\left((1, a_1, a_2, ..., a_{m-1})\right)_\mu\right] = \left((\underbrace{1, 2, 2, ..., 2}_{\leftarrow r \text{ twos} \rightarrow}, a_1, a_2, ..., a_{m-1})\right)_\mu$$

for any $\left((1, a_1, a_2, ..., a_{m-1})\right)_\mu \in M_P$, where r is a fixed arbitrary natural number $\ni r \le m-1$.

φ_{f_r} is a homomorphism on M_P.

Increasing class sizes:

Theorem 5: Let h_1, h_2, …,h_{m-1} be elements of $\{0, \pm 2, \pm 4, ...\}$ and let:

$$\Im^s_{P,m} = \{\left((1, a_1, a_2, ..., a_{m-1})\right) \in \Im_{P,m} \mid 2 \le a_1 + h_1 \le a_2 + h_2 \le .... \le a_{m-1} + h_{m-1}\}$$

Then $\left(\Im^s_{P,m} \le_s\right)$ is a sublattice of $\left(\Im_{P,m}, \le_s\right)$.

Proof: We prove that the g.l.b. and the l.u.b. of two arbitrary elements of $\Im^s_{P,m}$ are also in $\Im^s_{P,m}$.

Let A and B be two arbitrary elements of $\Im^s_{P,m} \ni$:

$$A = \left((1, a_1, a_2, ..., a_{m-1})\right)$$

And:

$$B = \left((1, b_1, b_2, ..., b_{m-1})\right)$$

$$A \vee B = ((1, \max\{a_1, b_1\}, \max\{a_2, b_2\}, ..., \max\{a_{m-1}, b_{m-1}\}))$$

Now, since $A, B \in \Im^s_{P,m}$:

$$2 \le a_1 + h_1 \le a_2 + h_2 \le ... \le a_{m-1} + h_{m-1}$$

And:

$$2 \le b_1 + h_1 \le b_2 + h_2 \le ... \le b_{m-1} + h_{m-1}$$
$$\therefore 2 \le \max\{a_1, b_1\} + h_1 \le \max\{a_2, b_2\} + h_2 \le ... \le \max\{a_{m-1}, b_{m-1}\} + h_{m-1}$$

$$A \vee B \in \Im^s_{P,m} \qquad (3)$$

Similarly:

$$A \wedge B \in \Im^s_{P,m} \qquad (4)$$

From (3) and (4), we see that $\left(\Im^s_{P,m}, \le_s\right)$ is a sublattice of $\left(\Im_{P,m}, \le_s\right)$.

Corollary 6: Let h_1, h_2,…h_{m-1} be elements $\{0, \pm 2, \pm 4, ...\}$:

$$M_{P,m} = \{\mu_P \mid P \in \Im_{P,m}\}$$

And:

$$M^s_{P,m} = \{\mu_P \mid P \in \Im^s_{P,m}\}$$

$$\left(M^s_{P,m} \le_\mu\right)$$

is a sublattice of $\left(M_{P,m}, \le \mu\right)$.

Theorem 7: Let $\Im^s_{P,m}, h_1, h_2, ..., h_{m-1}$ be as in the previous theorem.

Also, let $g : \Im^s_{P,m} \to \Im_{P,m}$ be defined by:

$$g\left[\left((1, a_1, a_2, ..., a_{m-1})\right)\right] = \left((1, a_1 + h_1, a_2 + h_2, ..., a_{m-1} + h_{m-1})\right)$$

for any element $\left((1, a_1, a_2, ..., a_{m-1})\right)$ of $\Im^s_{P,m}$.

g is a lattice homomorphism.

Proof: Clearly g is a function that maps elements of $\Im^s_{P,m}$ to elements of $\Im_{P,m}$.

We show that g is a homomorphism.

Let:

$$A, B \in \Im^{s}_{P,m} \ni$$

$$A = \left(\left(1, a_1, a_2, ..., a_{m-1}\right)\right)$$

And:

$$B = \left(\left(1, b_1, b_2, ..., b_{m-1}\right)\right)$$

$$\begin{aligned} g\left(A \vee B\right) &= g[((1, a_1, a_2, ..., a_{m-1})) \vee \left(\left(1, b_1, b_2, ..., b_{m-1}\right)\right)] \\ &= ((1, \max\left\{a_1, b_1\right\} + h_1, \max\left\{a_2, b_2\right\} + h_2, ..., \\ &\quad \max\left\{a_{m-1}, b_{m-1}\right\} + h_{m-1})) \\ &= \left(\left(1, a_1 + h_1, a_2 + h_2, ..., a_{m-1} + h_{m-1}\right)\right) \vee \\ &\quad \left(\left(1, b_1 + h_1, b_2 + h_2, ..., b_{m-1} + h_{m-1}\right)\right) \\ &= \left[g\left(A\right)\right] \vee \left[g\left(B\right)\right] \end{aligned}$$

Similarly, it can be shown that:

$$g\left(A \wedge B\right) = \left[g\left(A\right)\right] \wedge \left[g\left(B\right)\right]$$

Hence g is a lattice homomorphism on $\Im^{s}_{P,m}$.

Remark: If h_1, h_2,…h_{m-1} are non-negative, even integers and $h_1 \le h_2 \le ... \le h_{m-1}$, then $\Im^{s}_{P,m} = \Im_{P,m}$ and g is a lattice endomorphism.

Corollary 8: Let $h_1, h_2, ..., h_{m-1}, M_{P,m}$ and $M^{s}_{P,m}$ be as in Theorem 7 and Corollary 6.

Define:

$$\varphi_g : M^{s}_{P,m} \to M_{P,m}$$

By:

$$\varphi_g\left[\left(\left(1, a_1, a_2, ..., a_{m-1}\right)\right)_{\mu}\right] = \left(\left(1, a_1 + h_1, a_2 + h_2, ..., a_{m-1} + h_{m-1}\right)\right)_{\mu}$$

for any element $\left(\left(1, a_1, a_2, ..., a_{m-1}\right)\right)_{\mu}$ of $M^{s}_{P,m}$. φ_g is a lattice homomorphism.

Selecting classes:

Notation: Let:

$$\Im_{P,\ge e} = \left\{((1, a_1, a_2, ..., a_{m-1})) \in \Im_P \middle| m \ge e\right\}$$

where, $e \in \left\{2, 3, ...\right\}$.

Remark: Clearly $\Im_{P,\ge e}$ is a sublattice of $\Im_P$; for if $A, B \in \Im_{P,\ge e}$ then $A \vee B$ and $A \wedge B$ would each have at least e classes and so $A \vee B, A \wedge B \in \Im_{P,\ge e}$.

Theorem 9: Let i_1, i_2,…i_{r-1} be fixed, arbitrary, natural numbers $\ni i_1 \le i_2 \le ... \le i_{r-1}$ and let U be a mapping on:

$$\Im_{P,\ge i_{r-1}} \ni U[((1, a_1, a_2, ..., a_{m-1}))] = ((1, a_{i_1}, a_{i_2}, ..., a_{i_{r-1}}))$$
$$\forall \left(\left(1, a_1, a_2, ..., a_{m-1}\right)\right) \in \Im_{P,\ge i_{r-1}}$$

U is a homomorphism from $\Im_{P,\ge i_{r-1}}$ to $\Im_{P,r}$.

Proof: U is clearly a function.

Also, for any element:

$$\left(\left(1, a_1, a_2, ..., a_{m-1}\right)\right) \text{of } \Im_{P,\ge i_{r-1}}$$

$$U\left[\left(\left(1, a_1, a_2, ..., a_{m-1}\right)\right)\right] = \left(\left(1, a_{i_1}, a_{i_2}, ..., a_{i_{r-1}}\right)\right) \in \Im_{P,r}$$

Since:

$$a_{i_1} \le a_{i_2} \le .... \le a_{i_{r-1}}$$

$\therefore$ U is a function from $\Im_{P,\ge i_{r-1}}$ to $\Im_{P,r}$.

Now, let $A, B \in \Im_{P,\ge i_{r-1}} \ni$:

$$A = \left(\left(1, a_1, a_2, ..., a_{m'-1}\right)\right)$$
$$B = \left(\left(1, b_1, b_2, ..., b_{m''-1}\right)\right)$$

For numbers:

$$m', m'' \in \left\{i_{r-1}, i_{r-1} + 1, i_{r-1} + 2, ...\right\}$$
$$\ni m' \le m''$$

$$\begin{aligned} U\left(A \vee B\right) &= U([((1, a_1, a_2, ..., a_{m'-1})) \vee \left(\left(1, b_1, b_2, ..., b_{m''-1}\right)\right)] \\ &= U[((1, \max\left\{a_1, b_1\right\}, \max\left\{a_2, b_2\right\}, ..., \\ &\quad \max\left\{a_{m'-1}, b_{m'-1}\right\}, \max\left\{a_{m'-1}, b_{m'}\right\}, \\ &\quad \max\left\{a_{m'-1}, b_{m'+1}\right\}, ..., \max\left\{a_{m'-1}, b_{m''-1}\right\}))] \\ &= ((1, \max\left\{a_{i_1}, b_{i_1}\right\}, \\ &\quad \max\left\{a_{i_2}, b_{i_2}\right\}, ..., \max\left\{a_{i_{r-1}}, b_{i_{r-1}}\right\})) \\ &= \left(\left(1, a_{i_1}, a_{i_2}, ..., a_{i_{r-1}}\right)\right) \vee \left(\left(1, b_{i_1}, b_{i_2}, ..., b_{i_{r-1}}\right)\right) \\ &= \left[U\left(A\right)\right] \vee \left[U\left(B\right)\right] \end{aligned}$$

Similarly $U(A \wedge B) = [U(A)] \wedge [U(B)]$.

Hence U is a lattice homomorphism on $\Im_{P,\geq i_{r-1}}$.

Notation: Let $M_{P,\geq e} = \{\mu_P | P \in \Im_{P,\geq e}\}$ where $e \in \{2,3,...\}$.

Remark: $M_{P,\geq e}$ is a sublattice of M_P.

Corollary 10: Let $i_1, i_2, ..., i_{r-1}$ be as in Theorem 9 and let ψ_μ be a mapping on:

$$M_{P,\geq i_{r-1}} \ni \psi_\mu[((1,a_1,a_2,...,a_{m-1}))_\mu]$$
$$= \left(\left(1, a_{i_1}, a_{i_2}, ..., a_{i_{r-1}}\right)\right)_\mu \forall \left(\left(1,a_1,a_2,..,a_{m-1}\right)\right)_\mu \in M_{P,\geq i_{r-1}}$$

ψ_μ is a homomorphism from $M_{P,i_{r-1}}$ to $M_{P,r}$.

Reducing some class sizes:

Theorem 11: Let $t_1, t_2, ..., t_{s-1}$ be positive, even numbers $\ni t_1 \leq t_2 \leq ... \leq t_{s-1}$ and let f be a function on:

$$\Im_{P,\geq s} \ni f\left[\left(\begin{pmatrix}1,a_1,a_2,\\ ...,a_{m-1}\end{pmatrix}\right)\right] = (1,h_1,h_2,...,h_{s-1},a_s,a_{s+1},...,a_{m-1})$$
$$\forall \left(\left(1,a_1,a_2,...,a_{m-1}\right)\right) \in \Im_{P,\geq s}$$

where:

$$h_i = \begin{cases} a_i & \text{if } a_i \leq t_i \\ t_i & \text{if } a_i > t_i \end{cases}$$

for $i = 1,2,...,s-1; m \geq s$.

f is a lattice homomorphism from $\Im_{P,\geq s}$ to I, where I is the ideal:

$$\{((1,a_1,a_2,...,a_{m-1})) \in \Im_{P,\geq s} | a_i \leq t_i; i = 1,2,3,....,s-1\}$$

Of:

$$(\Im_{P,\geq s}, \leq_s)$$

Proof: f is clearly a function. Also for any two elements A and B of:

$$\Im_{P,\geq s} \ni A = \left(\left(1,a_1,a_2,...,a_{m-1}\right)\right)$$

$$B = \left(\left(1,b_1,b_2,...,b_{m'-1}\right)\right)$$

$(m' \geq m)$

$$\begin{aligned} f(A \vee B) &= f[((1,a_1,a_2,...,a_{m-1})) \vee ((1,b_1,b_2,....,b_{m'-1}))] \\ &= f[((1,\max\{a_1,b_1\},\max\{a_2,b_2\}, \\ &\quad ...,\max\{a_{m-1},b_{m-1}\},\max\{a_{m-1},b_m\}, \\ &\quad \max\{a_{m-1},b_{m+1}\},...,\max\{a_{m-1},b_{m'-1}\}))] \\ &= ((1,k_1,k_2,...,k_{s-1},\max\{a_s,b_s\},\max\{a_{s+1},b_{s+1}\}, \\ &\quad ....,\max\{a_{m-1},b_{m-1}\},\max\{a_{m-1},b_m\}, \\ &\quad \max\{a_{m-1},b_{m+1}\},....,\max\{a_{m-1},b_{m'-1}\})) \end{aligned}$$

Where:

$$k_i = \begin{cases} \max\{a_i,b_i\} & \text{if } \max\{a_i,b_i\} \leq t \\ t_i & \text{if } \max\{a_i,b_i\} > t_i \end{cases}$$

$(\text{for } i = 1,2,...,s-1)$

$$= ((1,u_1,u_2,....,u_{s-1},a_s,a_{s+1},...,a_{m-1})) \vee \left(\left(1,v_1,v_2,...,v_{s-1},b_s,b_{s+1},...,b_{m'-1}\right)\right)$$

Where:

$$u_i = \begin{cases} a_i & \text{if } a_i \leq t_i \\ t_i & \text{if } a_i > t_i \end{cases}$$

$(\text{for } i = 1,2,...,s-1)$

And:

$$v_i = \begin{cases} b_i & \text{if } b_i \leq t_i \\ t_i & \text{if } b_i > t_i \end{cases}$$

$(\text{for } i = 1,2,...,s-1)$

$$\therefore f(A \vee B) = [f(A)] \vee [f(B)]$$

Now:

$$\begin{aligned} f(A \wedge B) &= f\left[\left(\left(1,a_1,a_2,...,a_{m-1}\right)\right) \wedge \left(\left(1,b_1,b_2,...,b_{m'-1}\right)\right)\right] \\ &\quad f\left[\left(\begin{pmatrix}1,\min\{a_1,b_1\},\min\{a_2,b_2\}, \\ ....,\min\{a_{m-1},b_{m-1}\}\end{pmatrix}\right)\right] \\ &= ((1,l_1,l_2,...,l_{s-1},\min\{a_s b_s\},\min\{a_{s+1},b_{s+1}\},...., \\ &\quad \min\{a_{m-1},b_{m-1}\})) \end{aligned}$$

Where:

$$l_i = \begin{cases} \max\{a_i, b_i\} & \text{if } \min\{a_i, b_i\} \le t \\ t_i & \text{if } \min\{a_i, b_i\} > t_i \end{cases}$$

$$= (1, w_1, w_2, ..., w_{s-1}, a_s, a_{s+1}, ..., a_{m-1})) \wedge$$
$$((1, x_1, x_2, ..., x_{s-1}, b_s, b_{s+1}, ..., b_{m-1})),$$
$$= \left[f(A)\right] \wedge \left[f(B)\right]$$

Where:

$$w_i = \begin{cases} a_i & \text{if } a_i \le t_i \\ t_i & \text{if } a_i > t_i \end{cases}$$

$$(i = 1, 2, ..., s-1)$$

And:

$$x_i = \begin{cases} b_i & \text{if } b_i \le t_i \\ t_i & \text{if } b_i > t_i \end{cases}$$

Where:

$$(i = 1, 2, ..., s-1)$$

Corollary 12: Let $t_1, t_2, ..., t_{s-1}$, and I be as in Theorem 11. Also let:

$$I_\mu = \{\mu_P | P \in I\}$$

Define: $\psi \,\text{on}\, \mu_{P,\ge s}$ by:

$$\psi\left[\left((1, a_1, a_2, .., a_{m-1})\right)_\mu\right] = \left((1, h_1, h_2, ..., h_{s-1}, a_s, a_{s+1}, ..., a_{m-1})\right)_\mu$$
$$\forall \left((1, a_1, a_2, ..., a_{m-1})\right)_\mu \in M_{P,\ge s}$$

Where:

$$h_i = \begin{cases} a_i & \text{if } a_i \le t_i \\ t_i & \text{if } a_i > t_i \end{cases}$$

$$(i = 1, 2, ..., s-1); m \ge s.$$

Ψ is a lattice homomorphism from $M_{P,\ge s}$ to I_μ.

RESULTS AND DISCUSSION

We have shown that there exists various lattice homomorphisms from lattice of SK-partitions to lattices of SK-partitions and that similar relationships exist between lattices of measures defined in term of SK-partitions.

This study furthers the study of systems of measures and relationships between such systems.

CONCLUSION

Using the approach used in this study, it is possible to do further study of lattices of measures defined in terms of SK-partitions by investigating lattices of these partitions.

REFERENCES

1. Birkhoff, G., 1948. Lattice Theory. American Mathematical Society, Rhode Island.
2. Kaushik, M.L., 1978. Burst-error-correcting codes with weight constraints under a new metric. J. Cybernet., 8: 183-202. http://www.informaworld.com/index/776410891.pdf
3. Kaushik, M.L., 1979. Single error and burst error correcting codes through a new metric. J. Cybernet., 8: 1-15. http://www.informaworld.com/index/775712451.pdf
4. Kaushik, M.L., 1979. Random error detecting and burst error correcting codes under a new metric. Indian J. Pure Appl. Math., 10: 1460-1468. http://www.new.dli.ernet.in/rawdataupload/upload/insa/INSA_2/20005a66_1460.pdf
5. Maharan, D., 1982. Orthogonality measures: An example. Ann. Probab., 10: 879-880. http://projecteuclid.org/DPubS?service=UI&version=1.0&verb=Display&handle=euclid.aop/1176993803
6. Niederreiter, H. and N. Sookoo, 2000. Partial densities on the group of integers. Arch. Math. (Brno) Tomus, 36: 17-24. http://www.emis.de/journals/AM/00-1/sookoo2.ps.gz
7. Niederreiter, H. and N. Sookoo, 2002. Partial densities on locally compact abelian groups and uniformly distributed sequences. Monatsh. Math. 136: 243-247. DOI: 10.1007/s00605-001-0472-x
8. Schmidt, K.D., 1986. Decompositions of vector measures in Riesz spaces and Banach lattices. Proc. Edinburgh Math. Soc., 29: 23-39. DOI: 10.1017/S0013091500017375
9. Sookoo, N. and P. Chami, 2007. Measures on the quotient spaces of the integers. J. Math. Stat., 3: 188-195.