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Weyl's Theorem for Algebraically Class A(s, t) Operators

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Abstract: In this study, we show that Weyl's theorem holds for algebraically class A (s, t) operator acting on Hilbert space. We prove: (i) Weyl's theorem holds for f (T) for every f belongs to holomorphic function of spectrum of T; (ii) generalized Weyl's theorem holds for T; (iii) the spectral mapping theorem holds for the Weyl spectrum of T and for the essential approximate point spectrum of T.

Key words: Class A (s, t), SVEP, isoloid, Weyl's theorem

INTRODUCTION

Let H be an infinite dimensional Hilbert space and B (H) denote the algebra of all bounded linear operator acting on H. Let T be an operator whose polar decomposition is T= U|T|, where $|T|=\sqrt{T^*T}$. An operator T is said to be p-hyponormal, for $p\in(0,1)$, if $(T^*T)^p \geq (TT^*)^p$ by A. Aluthge. A 1-hyponormal operator is hyponormal and $\frac{1}{2}$- hyponormal operator is said to be semi-hyponormal. An invertible operator T is said to be log - hyponormal if $\log|T| \geq \log|T^*|$ (Tanahashi., 1999). An operator T is said to be paranormal if $\|T^2x\| \geq \|Tx\|^2$ for every unit vector x. As a generalization of a class A and paranormal operators they introduced class A(k) and absolute k-paranormal operator respectively (Furuta *et al*., 1998). An operator T belongs to class A (k) for k>0 if $(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2$ an absolute k-paranormal if $\||T|^k Tx\| \geq \|Tx\|^{k+1}$ for every unit vector $x\in H$. It has been proved that every log-hyponormal operator is class A (k), every class A(k) operator is absolute k-paranormal (Furuta *et al*.,1998). When k = 1 we say that T belongs to class A operator.

As a further generalization of class A (k), (Masatoshi *et al*., 2000) introduced the class A(s, t). An operator T belongs to class A (s, t), for s>0 and t>0 if $(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{t+s}} \geq |T^*|^{2t}$. In other words class A(s, t) operator is defined as $|T(s,t)|^{\frac{2t}{s+t}} \geq |T|^{2t}$. Many others have studied its properties in (Yamazaki *et al*., 1999; Ito and Yamazaki, 2002; Uchiyama, 2001) and (Uchiyama and Tanahashi, 2002). It is known that p-hyponormal operators and log - hyponormal operators are class A(s, t) operators and T (s, t) is $\frac{\min s,t}{s+t}$ - hyponormal for all 0<s, t. If T is a class A(s, t) operator and s≤s', t≤t', then T is a class A (s', t'). An operator T is a class A (1, 1) operator if and only if T is a class A operator (Masatoshi *et al*., 2000; Yamazaki *et al*., 1999; Wang and Lee, 2003; Ito, 1999; Ito and Yamazaki, 2002; Tanahashi, 1999; Yoshino, 1997). Class AI (s, t) is the class of all invertible class A(s, t) operator for s>0 and t>0. It was pointed out in (Yanagida, 2000) that class A (k, 1) equals class A (k). They showed several properties of class A (s, t) and class AI (s, t) as extensions of the properties of class A (k) shown in (Yamazaki *et al*., 1999). Spectral properties of class A (s, t) operators where s, t∈ (0, 1) have been studied by several authors ((Uchiyama and Tanahashi, 2002, Uchiyama *et al*., 2004). The spectral properties of class A (s, t) operators where s>1, t>1 via their generalized Aluthge transformation and hyponormal transforms has been studied by Stella. An operator T is said to be of algebraically class A (s, t), if there exists a non-constant complex polynomial p such that p (T) is of class A (s, t) operator.

Preliminaries:

Lemma 1: (Rashid and Zguitti, 2011): Let T belongs to the class A (s, t) for some 0<s, t≤1, $\lambda\in C$ and assume that σ (T) =λ. Then T =λ.

Lemma 2: Let T be invertible and quasi nilpotent algebraically class A (s, t) operator. Then T is nilpotent.

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Proof: Suppose that p (T) is class A (s, t) operator for some non-constant polynomial p. Since $\sigma(p(T)) = p(\sigma(T))$, the operator p(T)-p(0) is quasi-nilpotent, from Lemma 1 we have:

$$CT^m(T-\lambda_1)(T-\lambda_2)......................$$
$$(T-\lambda_n) \equiv p(T)-P(0)=0$$

where, $m\geq 1$ since $T-\lambda_i$ is invertible for every $\lambda_I \neq 0$ and so $T_m = 0$.

Lemma 3: Let T be an algebraically class A (s, t) operator. Then T is isoloid.

Proof: Let $\lambda \in is\sigma(T)$ and let $E_\lambda = \frac{1}{2\pi i}\int_{\partial D\lambda}(Z-T)^{-1}dz$ be the associated Riesz idempotent, where D_λ is a closed disc centered at λ which contains no other point of σ (T). We can represent T as the direct sum, $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ where $\sigma(T_1) = \lambda$ and $\sigma(T_2) = \sigma(T)\backslash\lambda$. Since Tis algebraically class A (s, t) operator p (T) is a class A (s, t) operator for some non-constant polynomial p. Since $\sigma(T_1) = \lambda$, we must have $\sigma(p(T_1)) = p(\sigma(T_1) = p(\lambda)$. Therefore $(p(T_1)) - p(\lambda)$ is quasi-nilpotent.

Since p (T_1) is class A (s, t) operator, it follows from Lemma 1, that p (T_1) –p (λ) = 0. Put. Then q (T_1) = 0 and hence T_1 is algebraically class A (s, t) operator. Since T_1-λ is quasi-nilpotent and algebraically class A (s, t) operator, it follows from Lemma 2, then T_1-λ is nilpotent. Therefore $\lambda \in \pi_0(T)$ and hence $\lambda \in \pi_0(T)$. This proves that T is isoloid.

Theorem 4 (Rashid and Zguitti, 2011): Let T belongs to the class A (s, t) for some $0<s, t\leq 1$. Then T is of finite ascent.

Corollary 5 (Rashid and Zguitti, 2011): Let T belongs to the class A (s, t) for some $0<s, t\leq 1$. Then T has SVEP.

Theorem 6: Let T be an algebraically class A (s, t) operator. Then T has SVEP.

Proof: First we show that if T is class A (s, t) operator, then T has SVEP. Suppose that T is class A (s, t) operator. If $\pi_0(T) = \varphi$, then clearly T has SVEP. Suppose that $\pi_0(T) \neq \varphi$. Let $\Delta(T)=\lambda \in \pi_0(T)$: $N(T-\lambda) \subseteq N(T^* - \overline{\lambda})$. Since T is class A (s, t) operator and $\pi_0(T) \neq \varphi$, $\Delta(T) \neq \varphi$ Let M be the closed linear span of the subspaces N (T-λ) with $\lambda \in \Delta$ (T). Then M reduces T and we can write T as $T_1 \oplus T_2$ on $H=M\oplus M^{\perp}$. Clearly T_1 is normal and $\pi_0(T_2) = \varphi$. Since T_1 and T_2 have both SVEP, T has SVEP. Suppose that T is algebraically class A (s, t) operator. Then p (T) is class A (s, t) operator for some non constant polynomial p. Since p (T) has SVEP, it follows from [Laursen and Neumann., 2000, Theorem 3.3.9] that T has SVEP.

RESULTS AND DISCUSSION

Weyl's theorem for algebraically class A(s,t) operators: Let $T\in B$ (H), we write N (T) and R (T) for the null space and range of T respectively. Let α (T) = dim N (T) and β (T) = dim N (T^*) where T^* is the adjoint of space of T. Let σ (T), $\sigma_p(T)$, $\sigma_\alpha(T)$ and π_0 (T), π_{00} (T) denote the spectrum, point spectrum, approximate point spectrum,the set of eigen values of T of finite multiplicity and isolated points of σ (T). An operator $T\in B$ (H) is called Fredholm if it has closed range, finite dimensional null space and its range has finite co-dimensional. The index of a Fredholm operator $T\in B$ (H) is given by:

$$ind(T) = \dim N(T) - \dim R(T)^{\perp} = (\dim N(T) - \dim N(T^*))$$

An operator $T\in B(H)$ is called Weyl if it is Fredholm of index zero. An operator $T\in B$ (H) is called Browder if it is Fredholm of "finite ascent and decent" equivalently (Harte, 1988) if T is Fredholm and $T-\lambda_i$ is invertible for sufficiently small $\lambda\neq 0$ in C. The essential spectrum $\sigma_p(T)$, the Weyl spectrum w(T) and the Browder spectrum $\sigma_b(T)$, of $T\in B$ (H) are defined in (Harte, 1985; 1988).

Theorem 7: Let T be an algebraically class A (s, t) operator. Then Weyl's theorem holds for T.

Proof: Suppose that $\lambda \in \sigma(T)\backslash w$ (T). Then T =λ is Weyl and not invertible, we claim that $\lambda \in \partial\sigma(T)$. Assume that λ is an interior point of $\sigma(T)$. Then there exist a neighborhood U ofλ, such that dim N (T-μ)>0 for all $\mu \in u$. It follows from [Finch, 1975 Theorem 10] that T does not have SVEP [single valued extension property]. On the other hand, since p (T) is class A (s, t) operator for some non constantpolynomial p, it follows from Lemma 6, that T has SVEP. It is a contradiction, Therefore $\lambda \in \partial\sigma(T)\backslash w$ (T) and it follows from the punctured neighborhood theorem that $\lambda \in \pi_{00}(T)$.

Conversely suppose that $\lambda \in \pi_{00}(T)$. Using the Riesz idempotent $E_\lambda = \frac{1}{2\pi i}\int_{\partial D\lambda}(Z-T)^{-1}dz$ for λ we can represent

T as the direct sum $T=\begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ where σ (T) = λ and σ (T_2) = σ (T)\λ.

Now we consider two cases: case (i)λ = 0.

Then T_1 is algebraically class A (s, t) operator and quasi nilpotent. It follows from Lemma 2, that T_1 is nilpotent. We claim that dim R (E) <∞. For if N (T_1) is infinite dimensional, then 0 does not belongs to $\pi_{00}(T)$. It is contradiction. Therefore T_1 is an operator on the finite dimensional space R (E). So it follows that T_1 is Weyl. But since T_2 is invertible, we can conclude that T is Weyl therefore $0\in \sigma(T)\setminus w(T)$.

Case (ii) λ≠0 Then by Theorem 3, T_1-λ is nilpotent. Since $\lambda\in\pi_{00}(T), T_1-\lambda$ is an operator on the finite dimensional space R (E). So T_1-λ is Weyl. Since T_2-λ is invertible, T-λ is Weyl.

By case (i) and case (ii), Weyl's theorem holds for T.This completes the proof.

Theorem 8: Let T be an algebraically class A (s, t) operator. Then Weyl's theorem holds for f (T) forevery $f\in H(\sigma(T))$.

Proof: Let $f\in H(\sigma(T))$. Since w (f (T)) ⊆ f (w (T)), it suffices to show that f (w (T)) ⊆ f (w (T)). Suppose $\lambda\notin w(f(T))$, then f (T)-λ is Weyl and Eq. 1:

$$f(T)-\lambda = C(T-\alpha_1)(T-\alpha_2)............(T-\alpha_n)g(T) \quad (1)$$

where, C, α1, α2,………., α∈C and g (T) is invertible. Since the operators in the rightside of (1) commute, every T-α_i is fredholm. Since T is algebraically class A (s, t) operator. T has SVEP by Lemma 6. It follows from (Aiena and Monsalve, 2000. Theorem 2.6] that ind (T-α_i) ≤0 for each i = 1,2,3,......n. Therefore λ∉f (T)) and hence f (w (T)) = (w (T)).

Now by (Lee and Lee, 1996), that is T is isoloid, then (σ(f (T))\$\pi_{00}$(T)) = σ(f (T))\$\pi_{00}$(T) for every f∈H (σ(T)). Since T is isoloid by Theorem 3 and Weyl's theorem holds for T by Theorem 7 σ(f (T))\$\pi_{00}$(T) = f (σ(T))\$\pi_{00}$(T) =f (w (T)) =w (f (T)) which implies that Weyl's theorem holds for f (T). This completes the proof.

Theorem 9: Let T be an algebraically class A (s, t) operator. Then generalized Weyl's theorem holds for (T).

Proof: Assume that λ∈σ(T)\σ_{Bw} (T). Then (T-λ_i) is B-Weyl and not invertible. We claim that λ∈∂σ (T). Assume to the contrary that λ is an interior point of σ (T). Then there exists a neighborhood U of λ such that dim (T-μ)>0 for all μ∈U. It follows from [Finch, 1975., Theorem 10] that T does not have SVEP. On the other hand since p (T) is class A (s, t) operator for non-constant polynomial p. It follows from Lemma 6 that p (T) has SVEP. Hence by [Laursen and Neumann, 2000, Theorem 3.3.9] T has SVEP, a contradiction. Therefore λ∈∂σ (T).Conversely, assume that λ∈E (T), then λ is isolated in σ (T). From [Koliha,1996, Theorem 7.1], we have X = M⊕N, where M, N are closed subspaces of X, U = (T-λI) $|$N is an invertible operator and $V=(A-\lambda I)|_N$ is a quasi - nilpotent operator. Since T is algebraically class A (s, t) operator, V is also algebraically class A (s, t) operator from Lemma 8, V is nilpotent. Therefore T-λI is Drazin invertible [Coburn, 1966, Proposition 19] and Lay, 1970, Corollary 2.2]. By [Berkani, 2002, Lemma 4.1]T-λI is a B - fredholm operator of index 0.

Theorem 10: Assume that T or T^* is algebraically class A (s, t). Then σ_{ea}(f (T)) =f (σ_{ea}(T)) for every f∈H (σ(T)).

Proof: Let f∈H (σ(T)). It suffices to that $f(\sigma_{ea}(T))\subseteq\sigma_{ea}(f(T))$ for every f∈H (σ(T)). Suppose that λ∉T (σ_{ea} (f (T))). Then f (T)-λ=C (T-α_1) (T-α_2)………. (T-α_n) g (T) where C, α_1, α_2,………,α_n and g (t) is invertible. If T is algebraically class A (s, t) operator, it follows from [Aiena and Monsalve, 2000, Theorem 2.6] that i (T-α_i) ≤0 for each i =1, 2, 3,........n. Therefore λ does not belongs to f (σ_{ea} (T)) and hence (σ_{ea} (T)) = f (σ_{ea} (T)).

Suppose that T^* is algebraically class A (s, t) then T^* is SVEP. Since i (T-α_i) ≤0 for each I = 1, 2, 3....n. (T-α_i) is Weyl for each I = 1, 2, 3,.......n. Hence λ∉f (σ_{ea} (T)) and so (σ_{ea} (T)) = f (σ_{ea} (T)).
This completes the proof.

In this study we discuss Weyls theorem holds for Class A (s,t).

CONCLUSION

It can be shown that that Weyl's theorem holds for algebraically class A(s, t) operator acting on Hilbert space H. It can also be shown that Weyl's theorem holds for f(T) for every $f\in H(\sigma(T))$ and generalized Weyl's theorem holds for (T). The spectral mapping theorem holds for the Weyl spectrum of T and for the essential approximate point spectrum of T is also shown.

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