

On Containment and Radical Graphs of a Commutative Ring

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Abstract: In this study, we consider various graphs with the vertices being sets of ideals of a commutative ring with identity, R . How we define the edge set will vary according to distinct ideal theoretic properties. We apply a graphical approach to the study of ideal relationships in commutative rings and from this we obtain some properties of their graphs.

Keywords: Commutative Ring, Radical Ideal, Connected Graph, Complete Graph

Introduction

The study of commutative ring theory using the concepts of graph theory has attracted considerable attention in the last years. Different graphs have been high-lighting the interplay between graph theory and algebraic structures. An example of such interplay can be found in the annihilating ideal graph. The authors in (Behboodi and Rakeei, 2011), defined the annihilating ideal graph to be a graph whose vertex set is the set of ideals with nonzero annihilators and there is an edge between I and J if and only if $IJ = 0$. In (Chakrabarty *et al.*, 2009), the authors introduced the intersection graph of ring in which vertices are non trivial ideals and there is an edge between I and J if and only if $I \cap J \neq 0$. In (Beck, 1988), Beck introduced the zero divisor graph where the set of vertices is the elements of R and two elements a and b in the ring R are adjacent if $ab = 0$. In (Anderson and Livingston, 1999), they introduced a refined definition of the zero divisor graph to give a good picture of the zero divisor structure of the ring. In the new definition the set of vertices is the set of nonzero zero divisors of the ring and two elements a and b in the ring R are adjacent if $ab = 0$. In (Ye and Wu, 2012) the author defined the comaximal ideal graph of a commutative ring R , whose vertices are the proper ideals of R which are not contained in the Jacobson radical of R and two vertices I and J are adjacent if and only if $I + J = R$.

In this study, we explore two types of graphs that appear when we use different assumptions to define our edge set where the vertex set of the graph is the set of the ideals of the commutative ring with identity R . This paper continues with the main target of research in the field of ideal graphs. Our particular focus is on Ω_1 , where edge structure is determined by containment and Ω_2 , the radical ideal graph which are obtained from the commutative ring with unity.

First, we would like to recall some basic definitions and results from graph theory (Chartrand, 2006) and ring theory (Dummit and Foote, 2004), which have been used through out the paper.

A graph $G = (V, E)$ consist of a finite nonempty set of vertices denoted by $V(G)$ and a collection of unordered pair of distinct vertices of G called edges denoted by $E(G)$. The cardinality of the vertex set of a graph G is called the order of G denoted by $|V|$. The degree of any vertex v of G is the number of edges incident with vertex v . In a graph G , a finite alternating sequence of vertices and edges starting and ending with vertices such that each edge in the sequence is incident with the vertices following and preceding it is called a Walk. A path is an open walk in which no vertex appears more than once. A graph G is connected if G contains a v_1 - v_2 path for every pair v_1, v_2 of distinct vertices of G . A graph G is complete if every two distinct vertices of G are adjacent.

We recall that an ideal P is prime ideal if $P \neq R$ and if $r, s \in P$ implies $r \in P$ or $s \in P$. An ideal M is a maximal ideal if $M \neq R$ and for any ideal $I \subseteq R$ with $M \subseteq I \subseteq R$ implies $M = I$ or $M = R$. We recall that an ideal is a radical if it is an intersection of primes, or equivalently, if $I = \sqrt{I} = \{r \in R \mid r^n \in I \text{ where } n \text{ is a positive integer}\}$.

Main Results

In this section, we introduce two types of graphs that appear when we use different assumptions to define our edge set, where the vertex set of the graph is the set of the ideals of the commutative ring with identity R and study its properties.

Definition 2.1

Let R be a commutative ring with unity. We define the graph Ω_1 if the vertex set is the set of the ideals of R

and two vertices I and J have an edge between them if and only if $I \subset J$.

By definition of our new graph, there is an edge between the zero ideal and any vertex in Ω_1 . Moreover, there is no loop in any vertex in Ω_1 .

Theorem 2.1

Ω_1 is connected graph with diameter less than or equal 2.

Proof

Of course, we have the path $I-(0)-J$.

Theorem 2.2

Ω_1 is complete if and only if R is an valuation ring.

Proof

Assume that Ω_1 is complete graph. Then any two vertices of the graph Ω_1 are connected. Let I and J be two ideals in R (vertices in the graph). Since (as vertices) I and J have an edge between them, then I and J (as ideals) are comparable. Hence R is a valuation ring.

Now we suppose that R is an valuation ring. Hence any two ideals are comparable and hence every two vertices are connected. hence, Ω_1 is complete.

Example 2.1

Consider the ring Z_{30} . The set of ideals in Z_{30} are (0) , (2) , (3) , (5) , (6) , (10) and (15) . Figure 1 shows the graph Ω_1 of the ring Z_{30} .

Of course, it is more interesting to ask questions concerning connectivity if (in the spirit of (Anderson and Livingston, 1999)) we modify our set of vertices for Ω_1 . In this sence, we define Ω_{1*} to be the subgraph of Ω_1 with vertices consisting of all nonzero ideals of R . We begin with a lemma.

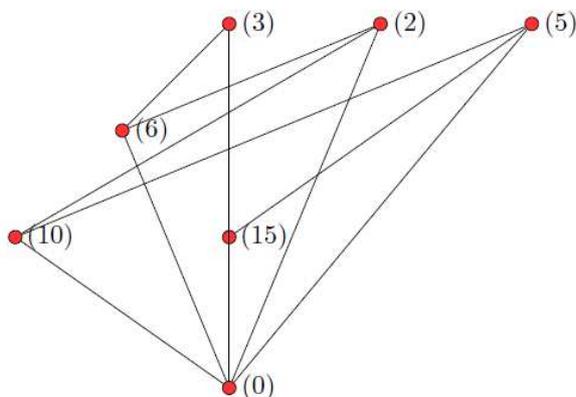


Fig. 1. $\Omega_1 (Z_{30})$

Lemma 2.1

Let R be commutative with identity and M_1, M_2, M_3 be distinct maximal ideals of R . If $M_1 M_2 = 0 = M_2 M_3$ then $M_2 = 0$.

Put loosely, this means that no maximal ideal can annihilate two distinct maximal ideals.

Proof

Since M_1 and M_3 are distinct maximal ideals, then we can find $m_1 \in M_1$ and $m_3 \in M_3$ such that $m_1 + m_3 = 1$. Multiplying this by a nonzero $m_2 \in M_2$, we obtain that $m_2 m_1 + m_2 m_3 = m_2 = 0$ which is our desired contradiction.

What follows is a couple of theorems to explain the structure (connectivity) of the graph Ω_{1*} . It is interesting that the case of two maximal ideals needs to be handled as a special case.

Theorem 2.3

Let R be a commutative ring with 1 and $\text{MaxSpec}(R) = M$ the collection of maximal ideals of R . The graph Ω_{1*} is connected if $|M| \neq 2$. Additionally $\text{diam}(\Omega_{1*}) \leq 2$ if $|M| = 1$ and $\text{diam}(\Omega_{1*}) \leq 6$ if $|M| > 2$.

Proof

Note that if $|M| = 1$ then there is a unique maximal ideal M and so given ideals $0 \neq I, J \subset R$, we have the path $I-M-J$. This also shows that the diameter does not exceed 2.

We now assume that there at least three maximal ideals. Assume that we have $0 \neq I, J \subset R$. We will assume that $I \subset M_1$ and that $J \subset M_2$. If $M_1 \cap M_2 \neq 0$, we select a nonzero element, a in this intersection and note that we have the path of length 4:

$$I = M_1 - (a) = M_2 - J$$

Now, we assume that $M_1 \cap M_2 = 0$. By the previous lemma, we now have that M_3 must have nontrivial intersection with both M_1 and M_2 . We therefore select $m_1 \in M_1 \cap M_3$ and $m_2 \in M_2 \cap M_3$ and this gives the following path of length 6:

$$I = M_1 - (m_1) = M_3 - (m_2) - M_2 - J$$

We now examine the outstanding case where $|M| = 2$.

Theorem 2.4

Let R be a commutative ring with 1 that has precisely two maximal ideals M_1 and M_2 , then Ω_{1*} is connected if and only if $M_1 M_2 \neq 0$ and in this case $\text{diam}(\Omega_{1*}) \leq 4$.

Proof

If $I \subset M_1, J \subset M_2$ and $0 \neq a \in M_1 M_2$, we have the path of length 4:

$$I - M_1 - (a) - M_2 - J$$

We now present a new graph that is specified by ideal theoretic properties and give some results for our new graph.

Definition 2.2

Let R be a commutative ring with unity. We define the graph Ω_2 if the vertex set is the set of ideals of R and two vertices I and J have an edge between them if and only if $\sqrt{I} = \sqrt{J}$.

Theorem 2.5

The connected components of the graph Ω_2 are in one to one correspondence with the distinct intersections of prime ideals in R .

Proof

It is well-known that an ideal is radical if and only if it is an intersection of prime ideals.

Corollary 2.1

Let R be a commutative ring of Krull dimension n . The number of connected components of Ω_2 is bounded below by $n + 1$ and if $\dim(R) = n < \infty$ then equality holds if and only if the prime ideals are linearly ordered.

Proof

If $P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_n$ is a chain of prime ideals then each P_i corresponds to a different connected component, which establishes the first statement.

If $\dim(R) = n$ then there exists a chain of prime ideals of R :

$$P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_n$$

If the primes are not linearly ordered then there is a prime ideal Q that does not compare to some P_i above. If this case the ideal $P_i \cap Q$ is a radical ideal that defines another connected component.

Corollary 2.2

Let R be a valuation domain of Krull dimension n . Then Ω_2 has precisely $n + 1$ connected components.

Proof

This follows from the proof of corollary 2.1.

Example 2.2

Consider the ring Z_{100} . The set of ideals in Z_{100} are (0) , (2) , (4) , (5) , (10) , (20) , (25) and (50) . By definition of the radical of an ideal I in R , its easy to

see that, $\sqrt{(2)} = (2)$, $\sqrt{(3)} = (3)$, $\sqrt{(4)} = (2)$, $\sqrt{(5)} = (5)$, $\sqrt{(10)} = (10)$, $\sqrt{(20)} = (10)$, $\sqrt{(25)} = (5)$, $\sqrt{(50)} = (10)$. It is clear that the zero ideal will not adjacent with any vertex in Ω_2 . Figure 2 shows the graph Ω_2 of the ring Z_{100} .

Theorem 2.6

Let R be a commutative ring with 1. The following conditions are equivalent:

1. Ω_2 is connected.
2. Ω_2 is complete.
3. R is quasilocal and $\dim(R) = 0$

Proof. (2) implies (1) is immediate as any complete graph is connected. For (1) implies (3), we note that if Ω_2 is connected then $\sqrt{I} = \sqrt{J}$ for all ideals in R . In particular $\sqrt{(0)}$ is the radical of every ideal of R . Hence R has a unique prime ideal and hence is quasilocal and of dimension 0. Finally, for (3) implies (2), we note that if R has a unique prime, then $\sqrt{I} = \sqrt{J}$ for all proper ideals $I, J \subset R$ and hence Ω_2 is complete.

We end with a last observation that we find interesting. By constructing appropriate valuation domains, we can create graphs Ω_2 with any arbitrary number of connected components, but such valuation domains (of dimension greater than 1) are necessarily non-Noetherian. The Noetherian case is somewhat more restrictive.

Theorem 2.7

Let R be Noetherian and suppose $\Omega_2(R)$ has only finitely many connected components. Then $\dim(R) \leq 1$ and R has only finitely many prime ideals. What is more:

1. If R has n prime ideals, then $\Omega_2(R)$ has no more than $2^n - 1$ connected components.
2. If R is an integral domain with n prime ideals then R has precisely $2^n - 1$ connected components.

Proof

It is well-known that if R is Noetherian of Krull dimension greater than 1, then R has in finitely many primes; this establishes the first statement.

To justify (1), we note that the distinct radical ideals consist of the distinct intersections of the n prime ideals. Counting the possible intersections by the number of ideals being intersected, we obtain:

$$\binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n - 1$$

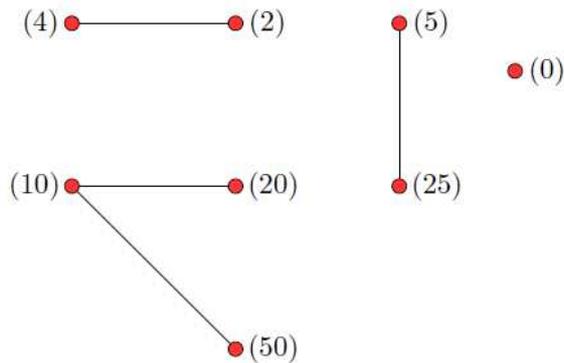


Fig. 2. $\Omega_2(Z_{100})$

This number of functions as the claimed are upper bound in (1).

If R is a domain, then (0) is a prime (and intersections involving (0) are, of course, trivial). If R is 0-dimensional then R is a field and the only prime (radical) ideal is (0) and hence the formula is verified. In the final case that $\dim(R) = 1$, then there are precisely $n-1$ nonzero primes and every distinct subset gives a unique radical ideal. Applying the formula from (1) and adding the ideal (0) , we obtain precisely $(2^{n-1}-1) + 1 = 2^{n-1}$ connected components.

Conclusion

In this study, we explored two types of graphs that appear when we use different assumptions to define our edge set, where the vertex set of the graph is the set of the ideals of the commutative ring with identity R . We concentrated on Ω_1 , where edge structure is determined by containment and Ω_2 , the radical ideal graph which are obtained from the commutative ring with unity. We gave some results and justifications for our investigated graphs. Also, we provided the reader with some examples that give a clear view of Ω_1 and Ω_2 .

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Ethics

This article is original and contains unpublished material. The corresponding author addresses any ethical issues that may arise after the publication of this manuscript.

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